Collins

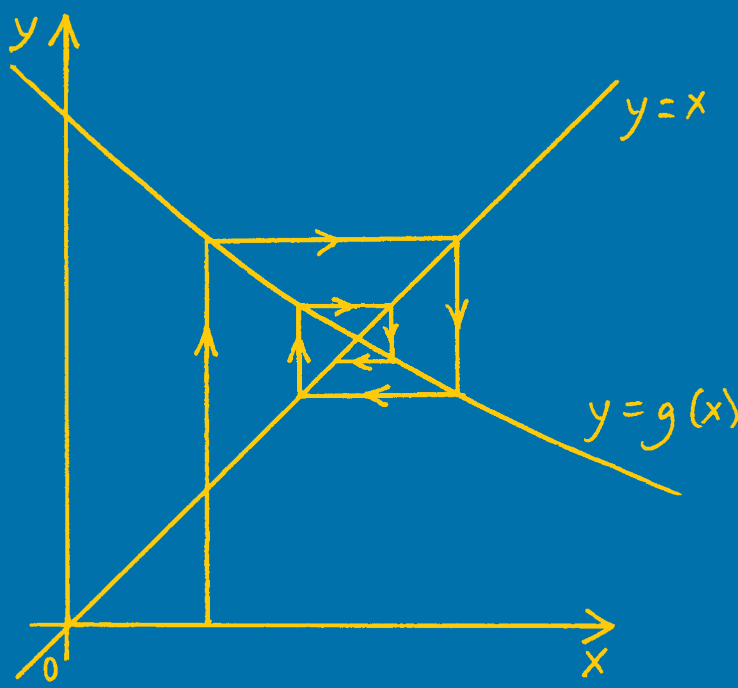
Cambridge Assessment International Education

Endorsed for Papers 2 and 3

Cambridge International AS & A Level Mathematics

Pure Mathematics 2 & 3

STUDENT'S BOOK



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1 ALGEBRA

Mathematics in life and work

This chapter focuses on manipulating algebraic expressions to make them simpler. This could involve finding the factors of an expression so that it can be fully factorised, or splitting a function containing a fraction into partial fractions so that the function can be used. The mathematics used in this chapter is widely applicable in a range of careers – for example:

- If you were a designer of road traffic systems, you would need to make the distinction between velocity and speed. Algebra can be used to find scalar quantities from vector quantities.
- If you were a financier, you might analyse financial transactions to determine the total value of money moved. Algebra could be used to find only positive numerical values.
- If you were an engineer working on the Large Hadron Collider, you would need to analyse the movement of particles. Algebra could be used to model changes in mass and energy over time.

In this chapter, you will consider the application of algebra to the design of road traffic systems.

LEARNING OBJECTIVES

You will learn how to:

- understand the meaning of $|x|$, sketch the graph of $y = |ax + b|$ and use relations such as $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x - a| < b \Leftrightarrow a - b < x < a + b$ in the course of solving equations
- divide a polynomial by a linear or quadratic polynomial, and identify the quotient and remainder
- use the factor theorem and the remainder theorem
- recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition
- use the expansion of $(1 + x)^n$, where n is a rational number and $|x| < 1$.

LANGUAGE OF MATHEMATICS

Key words and phrases you will meet in this chapter:

- absolute value, binomial expansion, divisor, factor theorem, modulus/moduli, partial fractions, polynomial, quotient, remainder, remainder theorem

PREREQUISITE KNOWLEDGE

You should already know how to:

- work with coordinates in all four quadrants
- recognise, sketch and interpret graphs of linear functions
- sketch transformations of a given function
- use and interpret algebraic manipulation
- use the concepts and vocabulary of expressions, equations, formulae, identities, inequalities, terms and factors
- simplify and manipulate algebraic expressions
- work out the coefficients in a binomial expansion.

You should be able to complete the following questions correctly:

1 Sketch the graphs of each of the following functions.

a $y = 2x + 5$ **b** $y = 3x - 2$ **c** $y = 3 - x$

2 Expand and simplify the following expressions.

a $(x + 1)(x + 2)$ **b** $(x - 3)(x + 4)$

c $(x - 2)(x - 3)$ **d** $(x - 3)^2$

e $(2x + 1)(x + 2)$ **f** $(x + 1)(x + 2)(x + 3)$

3 Find the coefficient of x^3 in a binomial expansion of $(2 + x)^6$.

1.1 The modulus of a linear function

The **modulus** of a number is its positive numerical value. The symbol for modulus is two vertical lines, one on either side of the function. For example, $|7| = 7$ and $|-7| = 7$. So you can see that any negative values are changed to be positive by the modulus.

On calculators, the modulus function is sometimes called **absolute value** so the button on your calculator may be labelled 'Mod' or 'Abs'.

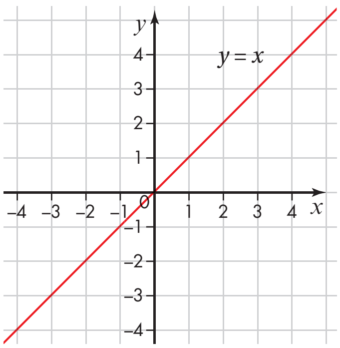
Generally, the modulus function is of the form $|x|$.

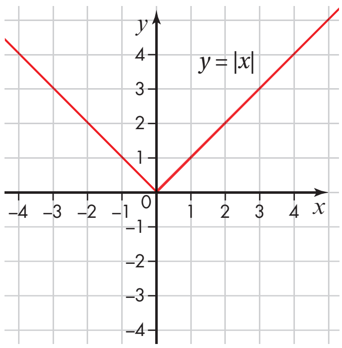
$|x| = x$ for $x \geqslant 0$

$|x| = -x$ for $x < 0$

So all the values of $|x|$ for $x < 0$ are positive.

For example, the graphs of $y = x$ and $y = |x|$ are:





KEY INFORMATION

The modulus function is of the form $|x|$.

- $|x| = x$ for $x \geqslant 0$
- $|x| = -x$ for $x < 0$

When graphing modulus functions of the form $y = |f(x)|$, an alternative approach to the algebraic method is to reflect in the x-axis any part of the graph that appears below the x-axis.

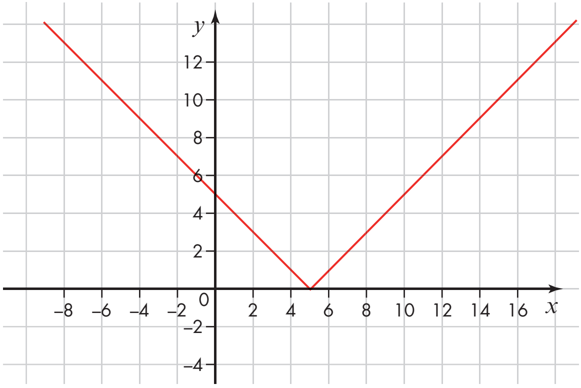
Example 1

Sketch the graph of $y = |x - 5|$.

Solution

First, sketch the graph of $y = x - 5$.

Any negative values from the function (negative y-values) will be changed to positive by the modulus, a reflection in the x-axis.

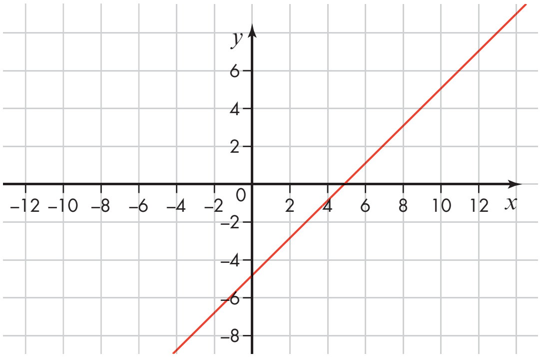


Example 2

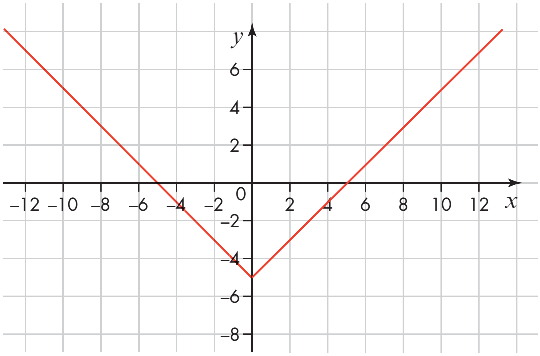
Sketch the graph of $y = |x| - 5$.

Solution

First you need to sketch the graph of $y = x - 5$ for $x \geqslant 0$.



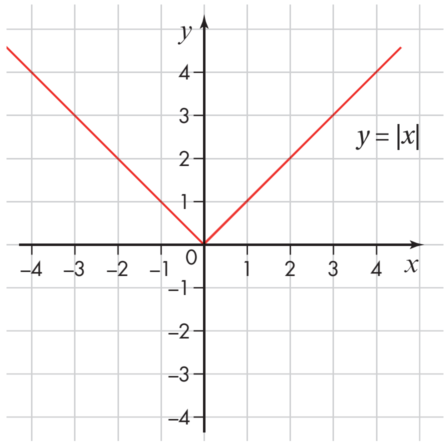
You then need to reflect the line in the y-axis.



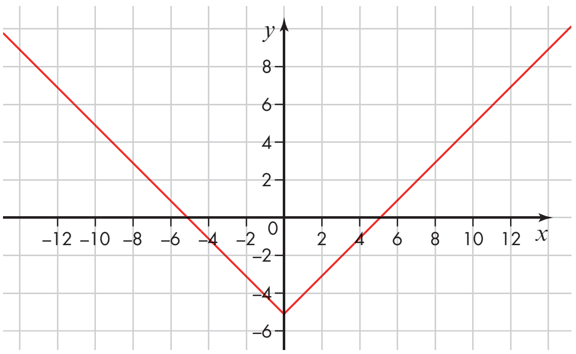
KEY INFORMATION

If you are not sure why the graph is reflected in the y-axis, create and compare the table of values for each of $y = x - 5$ and $y = |x| - 5$.

Alternatively you could draw the graph of $y = |x|$.



You then translate this graph by $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ to obtain the graph of $y = |x| - 5$.



Exercise 1.1A

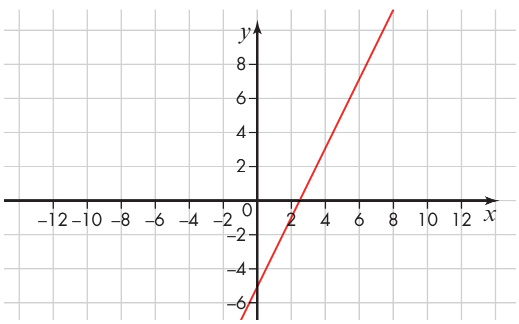
1 Sketch the graphs of each of the following functions on different sets of axes. Clearly show the coordinates of any axis intercepts.

a $y = |3x + 2|$ **b** $y = |x - 2|$

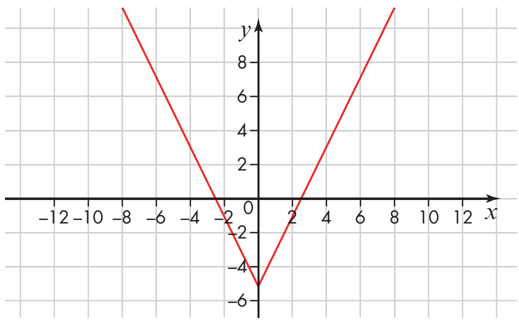
c $y = 3|x| + 2$ **d** $y = |x| - 2$

C **2** Which of the following is the graph of $y = |2x - 5|$? Clearly explain and justify your choice.

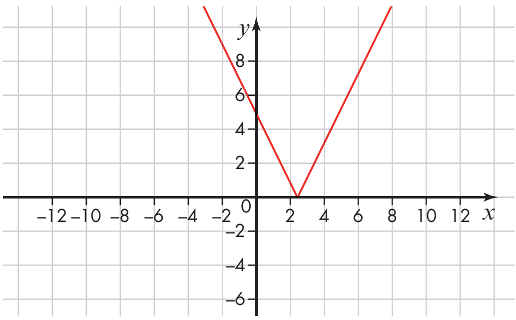
A



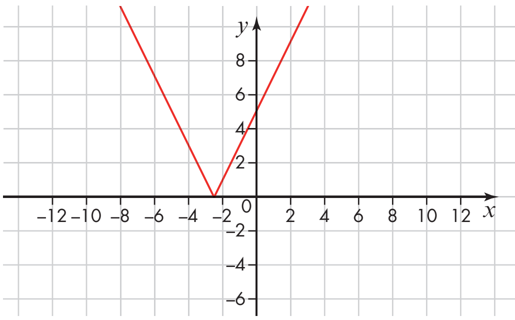
B



C



D

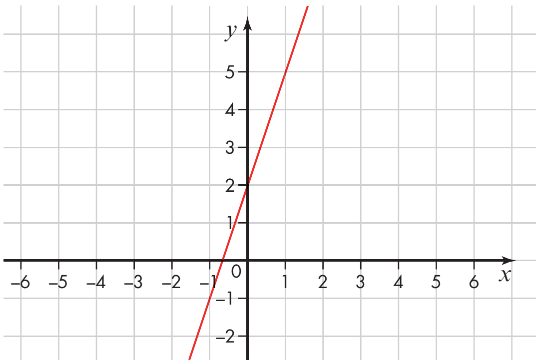


3 Sketch the graphs of each of the following functions on different sets of axes. Clearly show the coordinates of any axis intercepts.

a $y = |6 - x|$ **b** $y = |-x|$

c $y = 6 - |x|$ **d** $y = -|x|$

4 Here is the graph of $y = \mathrm{f}(x)$.

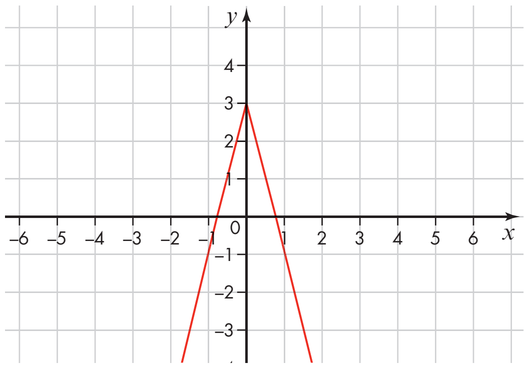


On separate sets of axes, sketch the graphs of each of the following functions. Clearly show the coordinates of any axis intercepts.

a $y = |\mathrm{f}(x)|$ **b** $y = \mathrm{f}(|x|)$

PS **5** Given that $\mathrm{f}(x) = 5 - 2x$, sketch the graph of $y = \mathrm{f}(|x|)$.

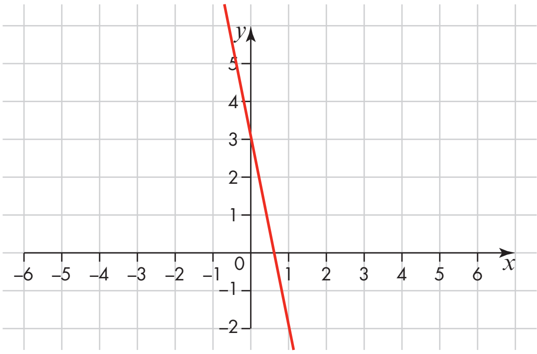
PS **6** Given that $\mathrm{f}(x) = 3 - 4x$, what transformation of this function would result in the function with the following graph?



7 **a** Given that $f(x) = 3 - 4x$, sketch the graph of $f(|x+2|)$.

b Describe the transformation from $f(x)$ to $f(|x+2|)$.

8 Here is the graph of $y = f(x)$.



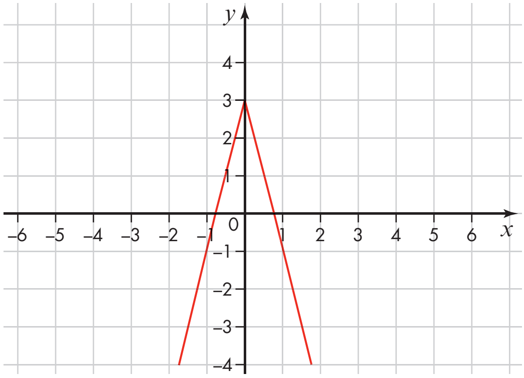
On separate pairs of axes, sketch the graphs of each of the following functions. Clearly show the coordinates of any axis intercepts.

a $y = f\left(\left|\frac{x}{2}\right|\right)$.

b $y = |f(x-2)|$.

c $y = |f(x)| + 3$.

PS **9** Given that $f(x) = 2x - 7$, detail the transformation(s) of this function that would result in the function with the following graph.



PS **10** Given that $f(x) = 3x + \frac{1}{2}$ and that A with coordinates $\left(\frac{1}{2}, 2\right)$ is a point on this line, find the coordinates of A when the following transformations are applied.

a $y = f(|2x|)$ **b** $y = |f(x-4)|$

c $y = |f(x)| + 5$

You need to be able to use the graphs of functions containing **moduli** to solve both equations and inequalities. For example, you need to be able to use the graph of $y = |2x - 1|$ to solve the equation $|2x - 1| = x$ or the inequality $|2x - 1| > x$.

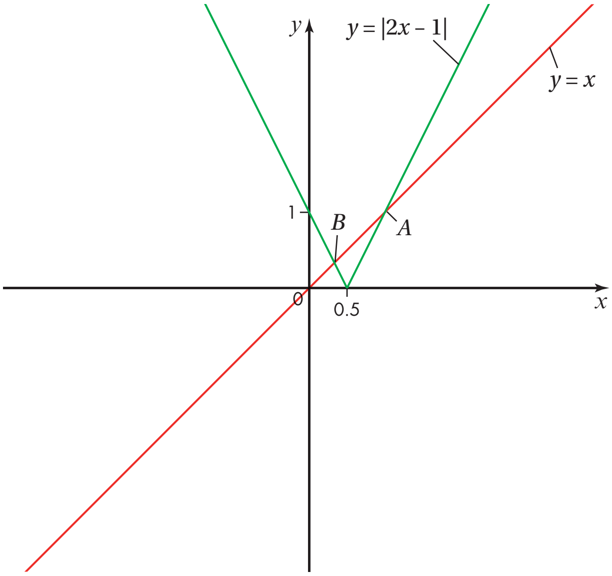
The plural of modulus is moduli.

Example 3

Solve the equation $|2x - 1| = x$.

Solution

On the same set of axes, sketch the graphs of $y = |2x - 1|$ and $y = x$.



There are two points of intersection, A and B.

At A: $2x - 1 = x$

Solving gives

$x = 1$

At B: $-(2x - 1) = x$

Solving gives $-2x + 1 = x$

$x = \frac{1}{3}$

An alternative algebraic method of solving this equation is to square both sides, which removes the modulus. You get:

$(2x - 1)^2 = x^2$

$4x^2 - 4x + 1 = x^2$

$3x^2 - 4x + 1 = 0$

$(3x - 1)(x - 1) = 0$

$x = \frac{1}{3}$ or $x = 1$