Collins

Cambridge International AS & A Level Further Mathematics

Further Pure Mathematics 1

STUDENT'S BOOK

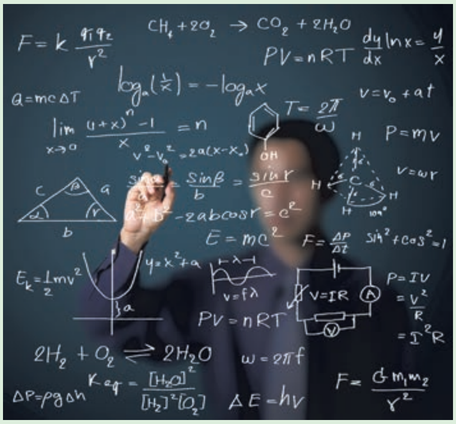
Tom Andrews, Helen Ball,
Chris Chisholm, Michael Kent
Series Editor: Dr Adam Boddison

1 ROOTS OF POLYNOMIAL EQUATIONS

Mathematics in life and work

In this chapter, the focus is on the relations between the coefficients of polynomials and their roots. One of our focuses is looking at how you could use the roots of one equation to find another equation with related roots. The mathematics used here is used in different careers. For example:

- If you were a scientist studying temperature of materials under certain conditions, you may be able to model your findings by using a polynomial.
- If you were an economist you might model the values of stocks and shares, using polynomials, to allow future predictions to be made. You could use the mathematics in this chapter to convert the predictions into different currencies.
- If you choose to study mathematics at university, the topics covered in this chapter will form an introduction to Galois theory. This theory can be used to show that there is no formula that can be used to find the roots of a polynomial of degree 5 and higher.



LANGUAGE OF MATHEMATICS

Key words and phrases you will meet in this chapter:

- cubic, pairwise product, quartic, root, symmetric function

LEARNING OBJECTIVES

You will learn how to:

- recall and use the relations between the roots and coefficients of polynomial equations
- use a substitution to obtain an equation with roots that are related in a simple way to those of the original equation.

PREREQUISITE KNOWLEDGE

You should already know how to…

- use the factor theorem
- use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs
- use the expansion of $(a + b)^n$
- add and multiply algebraic fractions.

You should be able to complete the following questions correctly:

1 Show that $(x - 2)$ is a factor of $f(x) = x^3 - 4x^2 + x + 6$.

2 Show that $(x - 3)$ is a factor of $f(x) = x^3 - 2x^2 - 23x + 60$.

3 Solve $x^2 + 4x + 5 = 0$.

4 Two roots of a cubic are $x = 2$ and $x = 3 + \mathrm{i}$. Find this cubic.

5 Expand $(x + 4)^3$.

6 Expand $(2x - 3)^4$.

7 Expand $\left(\frac{1}{x} + 5\right)^4$.

8 Simplify the following.

a $\frac{x}{2} + \frac{y}{4}$

b $\frac{x}{y} + \frac{y^2}{x}$

c $\frac{x^2y}{z} \times \frac{xz}{y}$

1.1 Roots of quadratic equations

In this section, you are going to be learning about the relations between the coefficients of a quadratic equation and its **roots**. When you understand these relations, you will be able to use them to solve equations with similar roots to the original.

Start by considering the quadratic equation $az^2 + bz + c = 0$ (where $a \neq 0$) which has roots α and β.

α is the Greek letter 'alpha' and β is 'beta'.

If α and β are roots of the equation then $(z - \alpha)$ and $(z - \beta)$ are factors of the equation. This means you can write this equation as $a(z - \alpha)(z - \beta) = 0$.

This gives the identity

$$\begin{aligned} az^2 + bz + c &\equiv a(z - \alpha)(z - \beta) \\ &\equiv a(z^2 - \beta z - \alpha z + \alpha\beta) \\ &\equiv az^2 - a(\alpha + \beta)z + a\alpha\beta \end{aligned}$$

Equate the coefficients of z:

$$b = -a(\alpha + \beta) \Rightarrow \alpha + \beta = -\frac{b}{a}$$

Equate the constant term:

$$c = a\alpha\beta \Rightarrow \alpha\beta = \frac{c}{a}$$

Note that this rule holds for both real and complex roots.

KEY INFORMATION

If a quadratic equation $az^2 + bz + c = 0$ (where $a \neq 0$) has factors $z - \alpha$ and $z - \beta$, then:

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

Example 1

Find a quadratic equation with roots –1 and 4.

Solution

The two roots of this equation are –1 and 4 so you can name one of them α and the other one β.

Let $\alpha = -1$ and $\beta = 4$.

In order to use the relations above, you need to know the value of $\alpha + \beta$ and $\alpha\beta$.

In this case, $\alpha + \beta = 3$ and $\alpha\beta = -4$

This means $\alpha + \beta = -\frac{b}{a} = 3$ and $\alpha\beta = \frac{c}{a} = -4$.

An alternative approach would be to use the fact that $(z + 1)$ and $(z - 4)$ are factors of the equation then expand them out, but the approach used in this example will make it easier to understand the technique before moving on to polynomials of a higher degree.

By substituting in different values for a, you are able to generate an infinite number of different quadratics. Substituting $a = 1$ can help keep the numerical calculations simple.

If $a = 1$ then you find that $b = -3$ and $c = -4$. If you substitute these values into the equation, you get that $z^2 - 3z - 4 = 0$ is a quadratic equation with roots –1 and 4.

Stop and think Write down three different quadratic equations with roots –1 and 4.

Example 2

Find a quadratic equation that has 3 + 2i as one of its roots.

Solution

You know from previous work on roots of polynomials that if one root is a complex number then its complex conjugate will also be a root. This means that if 3 + 2i is a root then 3 – 2i is also a root.

Let $\alpha = 3 - 2i$ and $\beta = 3 + 2i$.

You need to know the values of $\alpha + \beta$ and $\alpha\beta$. These are:

$\alpha + \beta = 3 - 2i + 3 + 2i = 6$

$\alpha\beta = (3 - 2i)(3 + 2i) = 9 - 6i + 6i - 4i^2 = 9 - 4(-1) = 13$

This means $\alpha + \beta = -\frac{b}{a} = 6$ and $\alpha\beta = \frac{c}{a} = 13$.

If you let $a = 1$ then $b = -6$ and $c = 13$.

So $z^2 - 6z + 13 = 0$ is a quadratic equation with roots 3 – 2i and 3 + 2i.

One application of the relations between the roots of one quadratic equation is to find a second quadratic equation with roots that are related to it. For example, you may want to find a quadratic equation with roots that are double those of a known one. There are two different methods for doing this: one using the relations and one using a substitution. Although the second method initially appears more time consuming than the first, it will become a very efficient technique when you work with polynomials of higher degree.

Suppose you have a quadratic with roots α and β. If you want to find the equation of a quadratic with roots 5α and 5β you can make a substitution into the original equation to find a new one.

To avoid any confusion with the variables, you should initially use a different variable to calculate the second quadratic. Here, you will use w.

The roots of this new quadratic are 5α and 5β so you can see that $w = 5\alpha$ is a solution to the new quadratic. This means that $z = \alpha = \frac{w}{5}$ is a solution of the original equation (because $z = \alpha$ is a root to the original quadratic). If you substitute $z = \alpha = \frac{w}{5}$ into the original equation you will get a new quadratic with the roots 5α and 5β in terms of w. Once you have this quadratic you can replace the variable w by z for consistency.

You will study these two techniques in the following example.

Example 3

The roots of $2z^2 + 7z - 2 = 0$ are α and β. Find a quadratic that has roots:

a 2α and 2β

b $3\alpha - 1$ and $3\beta - 1$.

Solution

a Method 1: using the relations

If $2z^2 + 7z - 2 = 0$ then $a = 2$, $b = 7$ and $c = -2$.

This means $\alpha + \beta = -\frac{b}{a} = -\frac{7}{2}$

and $\alpha\beta = \frac{c}{a} = -1$.

You want a quadratic with roots 2α and 2β. In order to use the relations to get the coefficients of the quadratics, you need to know the value for $(2\alpha + 2\beta)$ and $(2\alpha \times 2\beta)$ (the sum and the product of the new roots). To do this you need to write them in terms of $\alpha + \beta$ and $\alpha\beta$ so you can substitute the values of these from the original equation.

So $2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times -\frac{7}{2} = -7 = -\frac{b}{a}$

and $4\alpha\beta = 4 \times \alpha\beta = 4 \times -1 = -4 = \frac{c}{a}$.

These values of a, b and c refer to the new quadratic equation.

Putting $a = 1$ gives $b = 7$ and $c = -4$.

This gives the quadratic equation $z^2 + 7z - 4 = 0$.

Method 2: using a substitution

You want an equation with roots 2α and 2β so you need to let $w = 2\alpha$.

You would get the same results if you let $w = 2\beta$.

$w = 2\alpha \Rightarrow \alpha = \frac{w}{2}$

α is a root of the equation, so you can substitute $z = \alpha = \frac{w}{2}$ into the original equation.

$$2\left(\frac{w}{2}\right)^2 + 7\left(\frac{w}{2}\right) - 2 = 0$$

$$\frac{1}{2}w^2 + \frac{7}{2}w - 2 = 0$$

$$w^2 + 7w - 4 = 0$$

Multiply both sides by 2 to avoid fractions in the answer.

To ensure consistency with the question, rewrite the variable w as z.

$z^2 + 7z - 4 = 0$

b **Method 1: using the relations**

From **part a**, you know that $\alpha + \beta = -\frac{b}{a} = -\frac{7}{2}$ and

$\alpha\beta = \frac{c}{a} = -1$.

If the roots of the new equation are $3\alpha - 1$ and $3\beta - 1$ you need to know the values of $(3\alpha - 1) + (3\beta - 1)$ and $(3\alpha - 1)(3\beta - 1)$ in order to use the relations.

Writing them in terms of $\alpha + \beta$ and $\alpha\beta$ you get:

$(3\alpha - 1) + (3\beta - 1) = 3(\alpha + \beta) - 2$ and

$(3\alpha - 1)(3\beta - 1) = 9\alpha\beta - 3(\alpha + \beta) + 1$

Substituting in the values for $\alpha + \beta$ and $\alpha\beta$, you get:

$$3(\alpha + \beta) - 2 = 3 \times \frac{-7}{2} - 2 = -\frac{25}{2} = -\frac{b}{a}$$

$$9\alpha\beta - 3(\alpha + \beta) + 1 = 9(-1) - 3\left(\frac{-7}{2}\right) + 1 = \frac{5}{2} = \frac{c}{a}$$

Putting $a = 1$ gives $b = \frac{25}{2}$ and $c = \frac{5}{2}$

This gives the quadratic equation $z^2 + \frac{25}{2}z + \frac{5}{2} = 0$. To avoid fractions you can multiply both sides by 2, which gives $2z^2 + 25z + 5 = 0$.

Stop and think Why might selecting $a = 2$ have been a better option in this case?

Method 2: using a substitution

You want an equation with roots $3\alpha - 1$ and $3\beta - 1$ so you need to let $w = 3\alpha - 1$.

$w = 3\alpha - 1 \Rightarrow \alpha = \frac{w+1}{3}$

$z = \alpha$ is a root of the original equation, so

$$2\left(\frac{w+1}{3}\right)^2 + 7\left(\frac{w+1}{3}\right) - 2 = 0$$

$$\frac{2}{9}\left(w^2 + 2w + 1\right) + \frac{7}{3}(w+1) - 2 = 0$$

Multiply both sides of the equation by 9 to remove the fractions.

$$2(w^2 + 2w + 1) + 21(w + 1) - 18 = 0$$

$$2w^2 + 4w + 2 + 21w + 21 - 18 = 0$$

Expanding the brackets.

$$2w^2 + 25w + 5 = 0$$

Rewrite the variable as z instead of w to give the quadratic equation $2z^2 + 25z + 5 = 0$.

Example 4

The roots of the quadratic equation $z^2 + 5z - 2 = 0$ are α and β. By making use of a suitable substitution, find the equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Solution

You want an equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. You need to let $w = \frac{1}{\alpha}$.

$w = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{w}$

Since $z = \alpha$ is a root of the original equation, you get

$$\left(\frac{1}{w}\right)^2 + 5\left(\frac{1}{w}\right) - 2 = 0$$

$$\frac{1}{w^2} + \frac{5}{w} - 2 = 0$$

$$1 + 5w - 2w^2 = 0$$

Multiply both sides by w^2.

$$2w^2 - 5w - 1 = 0$$

Rewrite the variable as z instead of w to give the quadratic equation:

$$2z^2 - 5z - 1 = 0$$

In the next example you will use the relations to help find missing coefficients of a quadratic.

Example 5

The quadratic equation z^2+bz+c has roots α and β. Given that $\alpha+\beta=4$ and $\alpha^2+\beta^2=8.5$, find the values of b and c.

Solution

Using the relations:

$$\alpha+\beta=-\frac{b}{a}=4$$

and taking $a=1$ you get $b=-4$.

You also know that:

$$\alpha\beta=\frac{c}{a}$$

You are not given the value of $\alpha\beta$ but you are given the value of $\alpha^2+\beta^2$.

If you start with $(\alpha+\beta)^2$ you get:

$$(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$$

Rearranging gives:

$$\alpha\beta=\frac{1}{2}\left((\alpha+\beta)^2-\left(\alpha^2+\beta^2\right)\right)$$
$$=\frac{1}{2}\left((4)^2-8.5\right)$$
$$=3.75$$

This gives you $\frac{c}{a}=3.75$ so $c=3.75$.

So in the equation, $b=-4$ and $c=3.75$.

If the question asked for a quadratic with integer coefficients you could multiply by 4.

Exercise 1.1A

1 Using the relationships $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$, find the simplest quadratic equations with these pairs of roots.

a 4 and –2

b 6 and –1

c $2+i$, $2-i$

2 Express the sum and product of the following roots in terms of $\alpha+\beta$ and $\alpha\beta$.

a $\alpha+5$ and $\beta+5$

b $2\alpha+5$ and $2\beta+5$

c α^2 and β^2

d α^2+2 and β^2+2

e $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

f $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$