Further functions

33

Contents:

Opening problem

In a wildlife conservation program, some pairs of red squirrels and water voles were released onto an island. Their numbers were monitored for several years, allowing the research team to model the populations.

After x years, there were:

$$S = \frac{400}{1 + 9 \times 0.56^x} \text{ pairs of squirrels, and}$$

$$V = \frac{1000}{1 + 199 \times 0.27^x} \text{ pairs of water voles.}$$

Things to think about:

- **a** What do the graphs of these relationships look like?
- **b** What are the y-intercepts of these graphs, and what do these values represent?
- **c** When are there equal numbers of squirrels and voles?
- **d** When are there more squirrels than water voles?
- **e** What happens to the populations after a long period of time?

In this Chapter, we continue our study of functions, including **cubic functions** and functions which are unfamiliar to us.

We will also see how functions can be used to model real-world situations, and how information from their graphs can be interpreted in the real-world context.

A CUBIC FUNCTIONS [3.1, 3.2]

We have already seen that:

- a **linear** function has the form $y = ax + b$, where a, b are constants, $a \neq 0$
- a **quadratic** function has the form $y = ax^2 + bx + c$, where a, b, c are constants, $a \neq 0$.

These functions are the first two members of a family of functions called **polynomials**.

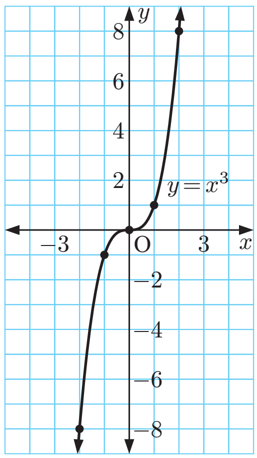
The next member of the polynomial family is the **cubic** function.

> A **cubic function** is a function of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c, and d are constants, $a \neq 0$.

The simplest cubic function is $y = x^3$. Its graph can be drawn from a table of values:

When $x = -2$, $y = (-2)^3 = -8$
When $x = -1$, $y = (-1)^3 = -1$
When $x = 0$, $y = 0^3 = 0$
When $x = 1$, $y = 1^3 = 1$
When $x = 2$, $y = 2^3 = 8$

x	-2	-1	0	1	2
y	-8	-1	0	1	8



Discovery — Cubic functions

To discover the shape of different cubic functions, you can either use the graphing package or your graphics calculator.

What to do:

1 **a** Use technology to help sketch, on the same set of axes:

   i $y = x^3$, $y = 2x^3$, $y = 3x^3$, and $y = \frac{1}{2}x^3$

   ii $y = x^3$ and $y = -x^3$

   iii $y = -x^3$, $y = -2x^3$, $y = -3x^3$, and $y = -\frac{1}{2}x^3$

   b Discuss the geometrical significance of a in $y = ax^3$. Comment on both the sign and the size of a.

2 **a** Use technology to help sketch, on the same set of axes:

$$y = x^3, \quad y = (x-2)^3 + 3, \quad y = (x-4)^3 - 2, \quad y = (x+3)^3 + 1$$

   b Discuss how the graph of $y = (x-h)^3 + k$ is related to the graph of $y = x^3$.

3 **a** Use technology to help sketch, on the same set of axes:

$$y = (x-4)(x-2)(x+1), \quad y = 2x(x-3)(x+2),$$
$$y = -x(x-2)(x+3), \quad y = -2(x-5)(x-1)(x+2).$$

   b Discuss the geometrical significance of α, β, and γ for the cubic $y = a(x-\alpha)(x-\beta)(x-\gamma)$.

4 **a** Use technology to help sketch, on the same set of axes:

$$y=(x-2)(x+1)(x+4), \qquad y=2(x-2)(x+1)(x+4),$$
$$y=\tfrac{1}{2}(x-2)(x+1)(x+4), \qquad y=-2(x-2)(x+1)(x+4).$$

b Discuss the geometrical significance of a in $y=a(x-\alpha)(x-\beta)(x-\gamma)$.

5 **a** Use technology to help sketch, on the same set of axes:

$$y=x(x+2)^2, \qquad y=(x+3)^2(x-1), \qquad y=2(x-1)^2(x+2),$$
$$y=\tfrac{1}{2}x^2(x+4), \qquad y=-2(x+2)(x-1)^2.$$

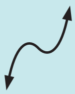
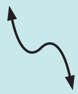
b Discuss the geometrical significance of α and β for the cubic $y=a(x-\alpha)^2(x-\beta)$.

6 **a** Predict the geometrical significance of a and α for the cubic $y=a(x-\alpha)^3$.

b Check your prediction is correct by sketching, on the same set of axes:

$$y=(x-2)^3, \quad y=-(x+1)^3, \quad y=2(x+3)^3, \quad y=-3(x+2)^3.$$

You should have discovered that:

- If $a>0$, the graph's shape is  or , if $a<0$ it is  or .
- $y=(x-h)^3+k$ is the translation of $y=x^3$ through $\begin{pmatrix} h \\ k \end{pmatrix}$.
- For a cubic function of the form $y=a(x-\alpha)(x-\beta)(x-\gamma)$, the graph has x-intercepts α, β, and γ, and the graph crosses over or **cuts** the x-axis at these points.
- For a cubic function of the form $y=a(x-\alpha)^2(x-\beta)$, the graph **touches** the x-axis at α and **cuts** it at β.
- For a cubic function of the form $y=a(x-\alpha)^3$, the graph cuts the x-axis at α. The curve changes shape at that point.

Example 1 — Self Tutor

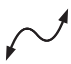
Use axes intercepts to sketch the graph of:

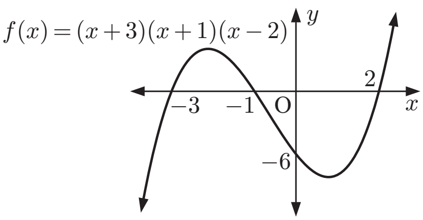
a $f(x)=(x+3)(x+1)(x-2)$ **b** $f(x)=-x(x+3)^2$

a $f(x)=(x+3)(x+1)(x-2)$ has x-intercepts -3, -1, and 2.

$f(0)=(3)(1)(-2)=-6$

$\therefore$ the y-intercept is -6.

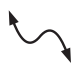
$a>0$ so the graph has shape .

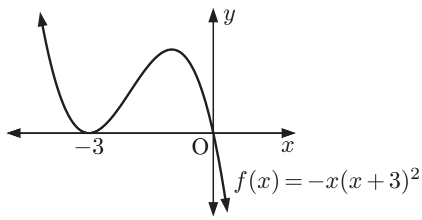


b $f(x)=-x(x+3)^2$ cuts the x-axis when $x=0$ and touches the x-axis when $x=-3$.

$f(0)=-0(3)^2=0$

$\therefore$ the y-intercept is 0.

$a<0$ so the graph has shape .



EXERCISE 33A

1 Determine whether each function is a cubic function:

a $y = 2x^3 + 3x^2 - x + 5$ **b** $y = x^3 - x^2 + \frac{1}{x}$ **c** $y = -x^3 - x + 7$

d $y = \frac{1}{2}x^3 + \frac{3}{4}x^2 - 1$ **e** $y = x^4 + 2x^3 - x + 3$ **f** $y = 4 - 5x + 3x^3$

2 Show that the following are cubic functions by expanding the brackets:

a $f(x) = (x+3)(x-2)(x-1)$ **b** $f(x) = (x+4)(x-1)(2x+3)$

c $f(x) = (x+2)^2(2x-5)$ **d** $f(x) = (x+1)^3 + 2$

3 Use axes intercepts to sketch the graph of:

a $f(x) = x(x-3)(x+2)$ **b** $y = (x-1)(x-4)(x+2)$

c $f(x) = -(x+3)(x-2)(x-4)$ **d** $y = 2x(x-1)(x+1)$

e $f(x) = -\frac{1}{2}(x+3)(x+1)(x-1)$ **f** $y = -3x(x+2)(x-1)$

4 Use axes intercepts to sketch the graph of:

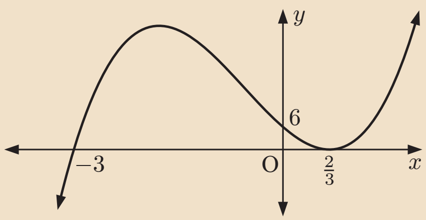
a $f(x) = (x-1)^2(x+1)$ **b** $y = -x(x+2)^2$ **c** $f(x) = -\frac{1}{2}(x-2)(x+2)^2$

d $y = \frac{1}{4}x^2(x+4)$ **e** $y = \frac{1}{3}(x-3)^3$ **f** $f(x) = -2(x+1)^3$

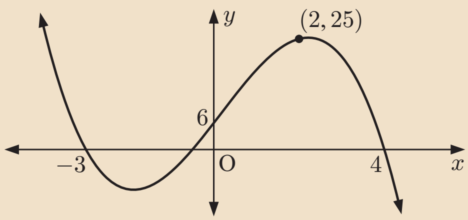
Example 2 — Self Tutor

Find the cubic function with graph:

a



b



a The graph *cuts* the x-axis at -3 and *touches* it at $\frac{2}{3}$.

$\therefore$ the function has the form $f(x) = a(x+3)(3x-2)^2$.

But $f(0) = 6$, so $a(3)(-2)^2 = 6$

$\therefore 12a = 6$

$\therefore a = \frac{1}{2}$

$\therefore f(x) = \frac{1}{2}(x+3)(3x-2)^2$

b The graph *cuts* the x-axis at -3 and 4.

We suppose the third linear factor is $(ax+b)$, so the function has the form $f(x) = (x+3)(x-4)(ax+b)$.

But $f(0) = 6$, so $(3)(-4)b = 6$

$\therefore -12b = 6$

$\therefore b = -\frac{1}{2}$

Also, $f(2) = 25$, so $(5)(-2)(2a - \frac{1}{2}) = 25$

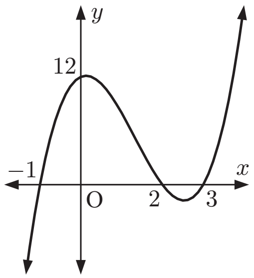
$\therefore 2a - \frac{1}{2} = -\frac{5}{2}$

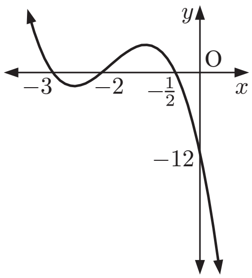
$\therefore 2a = -2$

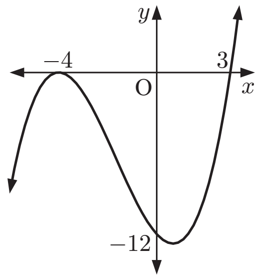
$\therefore a = -1$

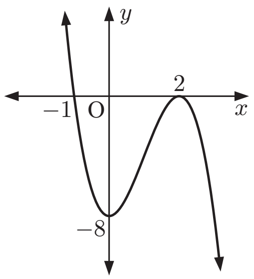
$\therefore f(x) = (x+3)(x-4)(-x-\frac{1}{2}) = -(x+3)(x-4)(x+\frac{1}{2})$

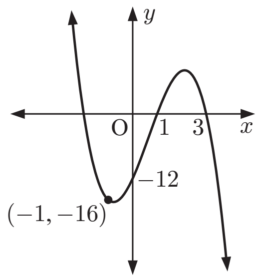
5 Find the cubic function with graph:

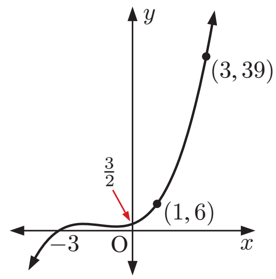
a


b


c


d


e


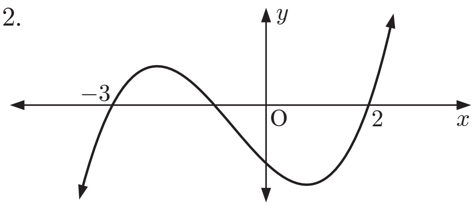
f


6 Find the equation of the cubic function which:

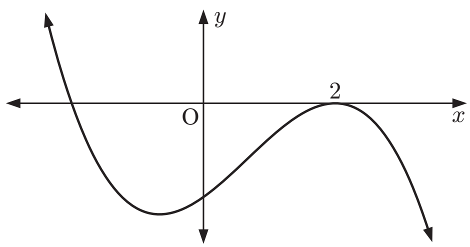
a has zeros 1 and 3, y-intercept 9, and passes through $(-1,\ 8)$

b touches the x-axis at 3, has y-intercept 18, and passes through $(1,\ 20)$.

7 The graph alongside has the form $y = 2x^3 + bx^2 + cx - 12$.
Find the values of b and c.



8 The graph alongside has the form
$y = -x^3 + bx^2 + 4x + d$.
Find the values of b and d.



B UNFAMILIAR FUNCTIONS [3.2]

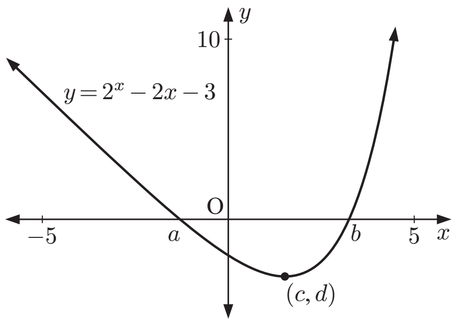
In this Section we practise using technology to study the features of functions that are unfamiliar to us.

When drawing a graph, remember to label the function, and record axes intercepts, turning points, and asymptotes.

GRAPHING PACKAGE

EXERCISE 33B

1 The graph of $y = 2^x - 2x - 3$ is shown alongside.



a Copy and complete this table of values:

x	-3	-2	-1	0	1	2	3
y							

b Find, to 3 significant figures, the value of:
i a **ii** b **iii** c **iv** d

2 For each of the following functions, use technology to sketch its graph. Clearly label axes intercepts, turning points, and asymptotes.

a $f(x) = 2 - \frac{3}{x+1}$
b $f(x) = 2x + \frac{1}{x}$
c $f(x) = 2^x - x$
d $f(x) = \frac{4x}{x^2 - 4x - 5}$
e $f(x) = \frac{x^2 - 1}{x^2 + 1}$
f $f(x) = \frac{x^2 + 1}{x^2 - 1}$
g $f(x) = \frac{2^x + 3}{2^x + 1}$
h $f(x) = x \times 2^{-x}$
i $f(x) = \sqrt{x^2 + 4}$
j $f(x) = x^2 - 3^x$
k $f(x) = \frac{2^x}{x^2}$
l $f(x) = 3^{-x^2}$

3 Consider $f(x) = x^3 - 4x^2 + 5x - 3$ with domain $-1 \leqslant x \leqslant 4$.

a Sketch the graph with the help of technology.
b Find the axes intercepts of the graph.
c Find and classify any turning points.
d State the range of the function.
e Create a table of values for $f(x)$ with x-steps of 0.5.

4 Consider $f(x) = x^4 - 3x^3 - 10x^2 - 7x + 3$ with domain $-2 \leqslant x \leqslant 1$.

a Create a table of values for $f(x)$ with x-steps of 0.25.
b Sketch the graph of the function. Use technology to help label the axes intercepts and turning points.

5 Consider the function $f(x) = \frac{12}{x+1} + x^2$.

a For what value of x is the function undefined? Explain your answer.
b Construct a table of values for $f(x)$ from $x = -3$ to $x = 3$ with x-steps of 1.
c Use technology to help sketch the graph of $y = f(x)$.
d Find the coordinates of the point where the curve crosses the x-axis.
e Find and classify the turning point.

C SOLVING EQUATIONS AND INEQUALITIES [2.5, 2.6, 3.2]

Whenever we have an equation or inequality which compares two expressions involving x, we can think of it as a comparison between two functions.

For example, to solve $x^2 - 2x = 3^x$ or $x^2 - 2x > 3^x$, we plot $y = x^2 - 2x$ and $y = 3^x$ on the same set of axes.

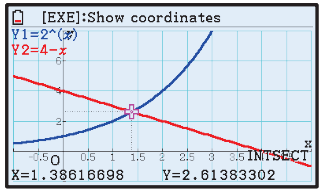
- The solutions of $x^2 - 2x = 3^x$ are the x-coordinates of the points where the graphs intersect.
- The solutions of $x^2 - 2x > 3^x$ are the values of x for which the graph of $y = x^2 - 2x$ is *above* the graph of $y = 3^x$.

Example 3 — Self Tutor

Solve for x, to 3 significant figures:

a $2^x = 4 - x$ **b** $2^x < 4 - x$

We graph $y = 2^x$ and $y = 4 - x$ on the same set of axes.

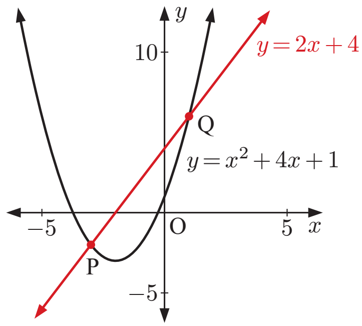


a The graphs intersect when $x \approx 1.39$.
$\therefore$ the solution is $x \approx 1.39$.

b $y = 2^x$ is *below* $y = 4 - x$ for $x < 1.39$.

EXERCISE 33C

1 The functions $y = x^2 + 4x + 1$ and $y = 2x + 4$ are graphed alongside.



a Use technology to find the coordinates of:
  i P **ii** Q

b Hence solve $x^2 + 4x + 1 = 2x + 4$.

c Check your answer to **b** using algebra.

d Use the graph to solve $x^2 + 4x + 1 \leqslant 2x + 4$.

2 **a** Use technology to help sketch the graphs of $y = x^3 - 3x + 5$ and $y = 7 - 2x$ on the same set of axes.

b Hence solve:
  i $x^3 - 3x + 5 = 7 - 2x$ **ii** $x^3 - 3x + 5 > 7 - 2x$

3 Solve using technology, giving your answers to 3 significant figures where necessary:

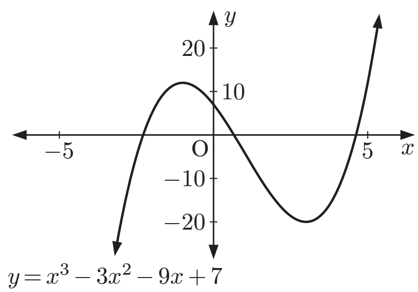
a $-2x^2 = x - 3$ **b** $\frac{1}{2}x^2 = x + 2$ **c** $(x+2)(x-1) = 2 - 3x$

d $(2x+1)^2 = \frac{1}{x^2}$ **e** $2^x = 3x$ **f** $\sqrt{x} = 3 - x$

g $x^2 = \sqrt{x+2}$ **h** $3^x = x^2$ **i** $x^3 + 2 = 3x - x^2$

j $x^3 - x + 3 = 0$ **k** $3^x = 5$ **l** $x^2 - 3 = \sqrt[3]{x}$

4 The function $y = x^3 - 3x^2 - 9x + 7$ is graphed alongside.



a Find the coordinates of:
  i the local maximum **ii** the local minimum.

b Solve the equation $x^3 - 3x^2 - 9x + 7 = 15$.

c Suppose the equation $x^3 - 3x^2 - 9x + 7 = k$ has exactly three solutions. What possible values could k have?

5 For what values of k does $\dfrac{x^2+4}{x^2+1} = k$ have exactly two solutions?

6 Solve for x:

a $x^2 + 2x - 3 > \dfrac{4}{x}$ **b** $5^x - 1 \leqslant x^3$ **c** $\dfrac{5}{x} < \dfrac{1}{\sqrt{x}} + 1$ **d** $2^x - 1 \geqslant \dfrac{1}{x^3}$

D PROBLEM SOLVING [2.5, 2.6, 3.2]

Real-world situations are not always modelled by linear or quadratic functions that we can easily manipulate using algebra. When more complicated functions are used, we can often use the graph of the function to answer questions about the real-world situation.

Example 4 Self Tutor

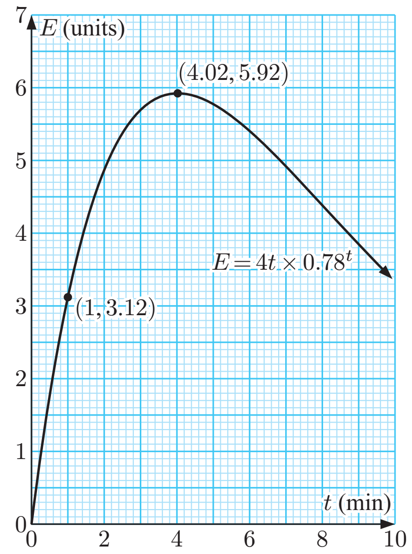
When a pain killing injection is administered, the effectiveness of the injection after t minutes is given by $E = 4t \times 0.78^t$ units.

a Sketch the graph of E against t for $t \geqslant 0$.

b Find the effectiveness of the injection after 1 minute.

c Find the maximum effectiveness of the injection, and the time when it occurred.

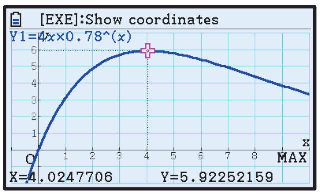
a



b When $t = 1$, $E = 4(1) \times 0.78^1 = 3.12$

So, the effectiveness of the injection after 1 minute was 3.12 units.

c



Using technology, there is a local maximum at $(4.02, 5.92)$.

So, the maximum effectiveness of the injection was about 5.92 units which occurred after about 4.02 minutes.

EXERCISE 33D

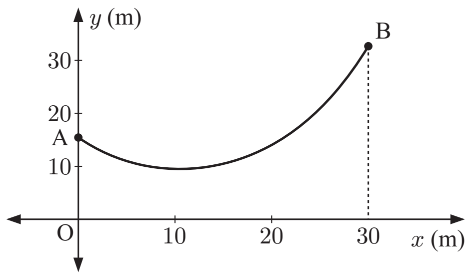
1 An enclosed box has length x cm, height 5 cm, and volume 400 cm^3. The surface area of the box is given by $y = 160 + 10x + \dfrac{800}{x}$ cm^2.

a Sketch the graph of this function for $0 \leqslant x \leqslant 20$.

b Find the surface area of the box if the length is 16 cm.

c Find the minimum surface area the box could have, and the length of box which produces it.

2 A chain is hung between two platforms at A and B. The height of the chain above ground level is given by $y = 1.84 \times 1.1^x + 13.6 \times 0.9^x$.



a Find the height of the platform at:
  i A **ii** B

b Find the height of the chain at its lowest point.

3 A new drug is undergoing trials to test its effectiveness. The amount in a patient's bloodstream t hours after administration is modelled by $A = 25t \times 0.641^t$ units.

a Find the amount of the drug in the patient:
  i initially **ii** after 2 hours **iii** after 5 hours.

b Use technology to help sketch the graph of A against t.

c When is the maximum amount of the drug in the patient?

d The patient feels the effects of the drug when its amount is at least 5 units. During what interval will the patient feel the effects of the drug?

4 One of the attractions at a theme park is a water slide. At the point where a person has travelled x m horizontally, their height above the end pool is given by $H = -\frac{x^3}{500} + \frac{97x^2}{1000} - \frac{3x}{2} + 15$ metres.

a Use technology to sketch the function from $x = 0$ until the slide reaches the end pool.

b Find the height of the starting platform above the end pool.

c How far horizontally does a person travel before they reach the end pool?

d What feature of the graph tells you when a person is sliding uphill or downhill?

e For what values of x is a person sliding uphill?

5 Answer the **Opening Problem** on page **607**.

6 Eve is standing at ground level. Her friend Mario is standing on a balcony 5 m above her. Simultaneously, they each throw a ball into the air. The heights of the balls above ground level after t seconds are given by the functions
$f(t) = -4.9t^2 + 12.5t + 6.5$ m and $g(t) = -4.9t^2 + 16t + 1.5$ m.

a Identify which ball corresponds to each function. Explain your answer.

b Use technology to sketch the functions on the same set of axes.

c Find the maximum height reached by each ball.

d At what time are the balls the same height above ground level? Discuss whether each ball is on its way up or down at this time.

e **i** Whose ball hits the ground first?
  ii How long after the first ball hits the ground does the second ball hit the ground?

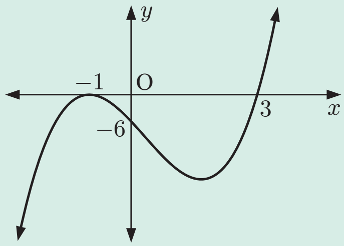
Review set 33A

1 Use axes intercepts to sketch the graph of:

a $y = x(x-2)(x+3)$ **b** $y = -2(x+1)^2(x-3)$

2 Find the cubic function which has x-intercepts -1, 0, and 2, and which passes through $(1, 6)$.

3 Find the equation of the cubic function with the graph alongside.



4 For each of the following functions, use technology to sketch its graph. Clearly label axes intercepts, turning points, and asymptotes.

a $f(x) = x^3 - 3^x$ **b** $f(x) = x^2 - \dfrac{1}{x}$ **c** $f(x) = \dfrac{5x}{x^2+3}$

5 Consider $f(x) = \dfrac{x^4}{5} - 3x^2 - 10$ with domain $-5 \leqslant x \leqslant 5$.

a Create a table of values for $f(x)$ with x-steps of 0.5.

b Sketch the graph of the function.

c Find the point where the graph crosses the y-axis.

d Find the zeros of $f(x)$.

e Find the coordinates of each local minimum.

6 **a** Use technology to help sketch the graphs of $y = 2^{-x}$ and $y = 5 - x^2$ on the same set of axes.

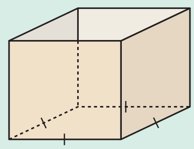
b Hence solve:

i $2^{-x} = 5 - x^2$ **ii** $2^{-x} \leqslant 5 - x^2$

7 Solve using technology:

a $x^3 = 2x - 3$ **b** $\sqrt{x} = 8 - x$ **c** $3^x > \sqrt{x^2+5}$

8 A crate is to be constructed with a square base x m long, an open top, and volume 6 m^3.



a Explain why:

i the height of the crate is $\dfrac{6}{x^2}$ m

ii the outer surface area of the crate is $\left(x^2 + \dfrac{24}{x}\right)$ m^2.

b Sketch the graph of $y = x^2 + \dfrac{24}{x}$ for $0 \leqslant x \leqslant 6$.

c Find the base side length which minimises the outer surface area of the crate.

Review set 33B

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