17

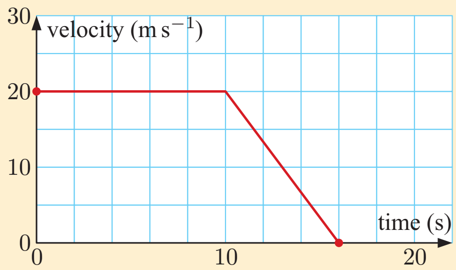
Applications of integration

Contents:

© HAESE MATHEMATICS - SAMPLE

Opening problem

This graph shows the velocity of a Formula One race car as it enters the pit lane.



Things to think about:

a For the first 10 seconds:

- **i** how *fast* does the car move
- **ii** how *far* does the car travel
- **iii** what is the area under the velocity graph?

b In total, how far did the car travel in the pit lane?

c Suppose we are given the velocity function $v(t)$ of a moving object. How can we determine the distance the object travels in a particular time interval?

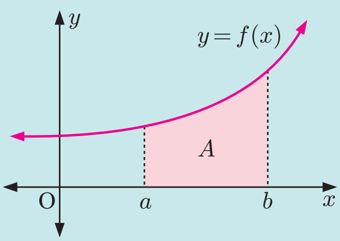
In this Chapter we further explore the relationship between integration and area, and consider other applications of integral calculus including kinematics.

A THE AREA UNDER A CURVE

We have already established in **Chapter 16** that:

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area A bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{where } F(x) = \int f(x)\,dx.$$



Example 1

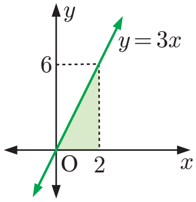
Self Tutor

Find the area of the region enclosed by $y = 3x$, the x-axis, $x = 0$, and $x = 2$ by using:

a a geometric argument

b integration.

a



$$\text{Area} = \tfrac{1}{2} \times 2 \times 6 = 6 \text{ units}^2$$

b

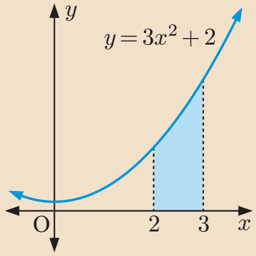
$$\begin{aligned}\text{Area} &= \int_0^2 3x\,dx \\ &= \left[\tfrac{3}{2}x^2\right]_0^2 \\ &= \tfrac{3}{2}(2)^2 - \tfrac{3}{2}(0)^2 \\ &= 6 \text{ units}^2\end{aligned}$$

© HAESE MATHEMATICS - SAMPLE

Example 2

Self Tutor

Find the shaded area.



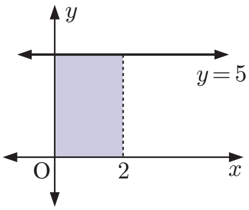
$$\begin{aligned}\text{Area} &= \int_2^3 (3x^2 + 2)\, dx \\ &= \left[x^3 + 2x\right]_2^3 \\ &= (27 + 6) - (8 + 4) \\ &= 21 \text{ units}^2\end{aligned}$$

EXERCISE 17A.1

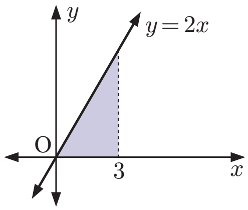
1 Find each shaded area using:

i a geometric argument **ii** integration.

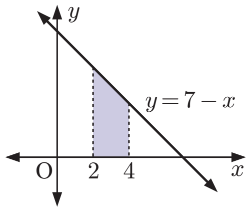
a



b

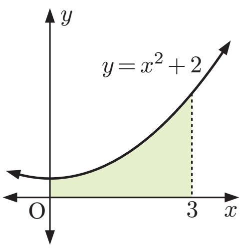


c

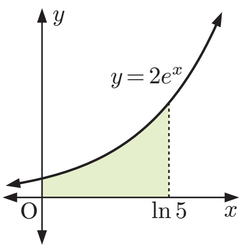


2 Find the shaded area:

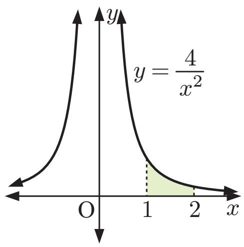
a



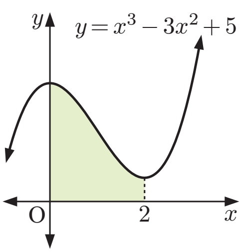
b



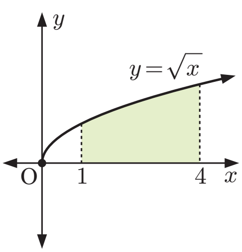
c



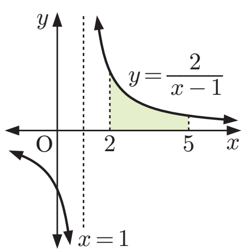
d



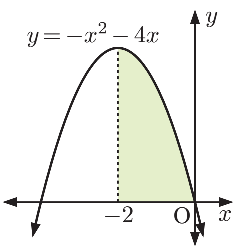
e



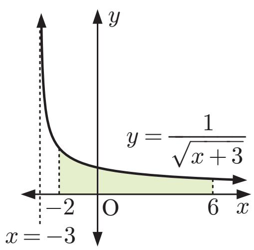
f



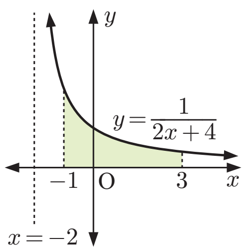
g



h

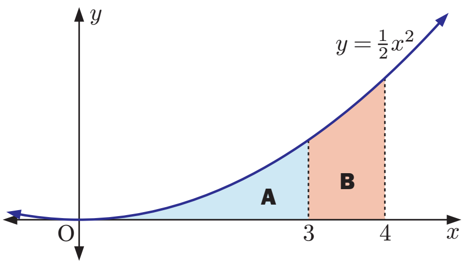


i



© HAESE MATHEMATICS - SAMPLE

3 Which of the shaded regions is larger?



4 Find the area of the region bounded by:

a $y = e^{2x}$, the x-axis, $x = 1$, and $x = 4$

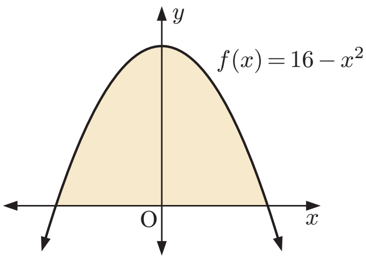
b $y = \cos x$, the x-axis, $x = 0$, and $x = \frac{\pi}{2}$

c $y = \dfrac{2}{5-x}$, the x-axis, $x = 2$, and $x = 4$

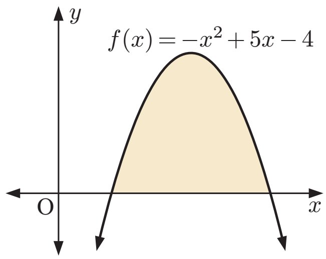
d $y = \sqrt{20-x}$, the x-axis, $x = 4$, and $x = 11$.

5 Find the shaded area:

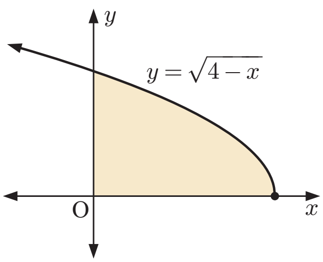
a



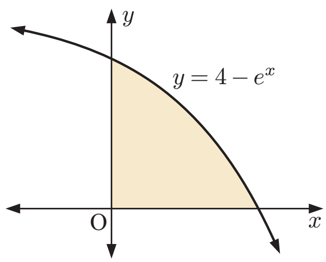
b



c



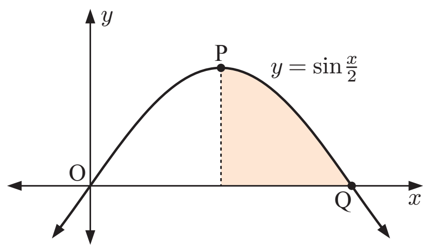
d



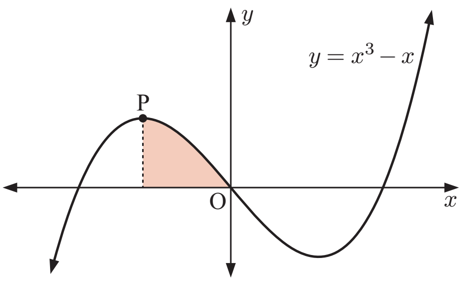
6 The graph of $y = \sin \frac{x}{2}$ shown has a maximum turning point at P, and cuts the x-axis at Q.

a Find the coordinates of P and Q.

b Find the shaded area.



7



The graph of $y = x^3 - x$ shown has a maximum turning point at P.

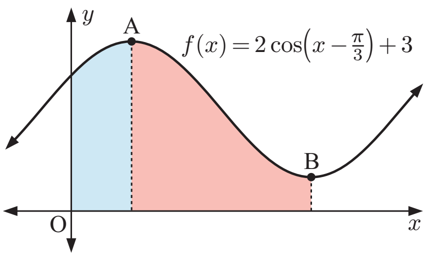
a Find the coordinates of P.

b Find the shaded area.

© HAESE MATHEMATICS - SAMPLE

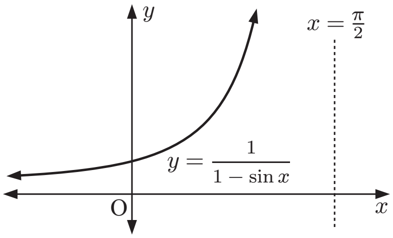
8 In the graph of $f(x) = 2\cos\left(x - \frac{\pi}{3}\right) + 3$ shown, A is a maximum turning point and B is a minimum turning point. Find the area of:

a the blue region **b** the red region.

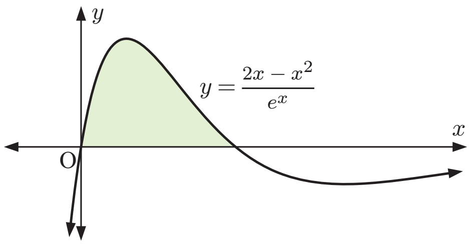


9 **a** Show that $\dfrac{d}{dx}\left(\dfrac{\cos x}{1-\sin x}\right) = \dfrac{1}{1-\sin x}$.

b Hence find the area bounded by the graph of $y = \dfrac{1}{1-\sin x}$ and the x-axis, between $x = 0$ and $x = \frac{\pi}{4}$.



10

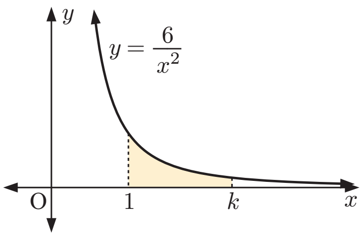


a Show that $\dfrac{d}{dx}\left(\dfrac{x^2}{e^x}\right) = \dfrac{2x - x^2}{e^x}$.

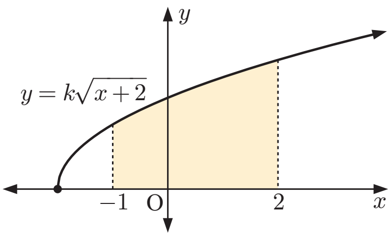
b Hence find the shaded area.

11 Find k such that:

a the shaded area is 4 units^2



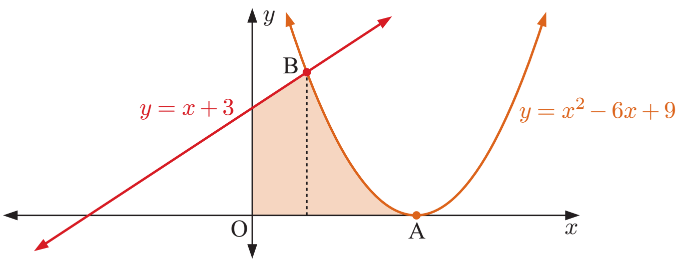
b the shaded area is 7 units^2.



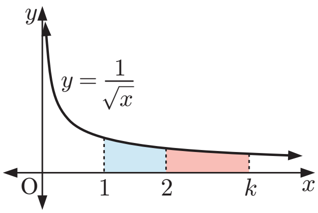
12 **a** Find the coordinates of:

i A **ii** B

b Find the shaded area.



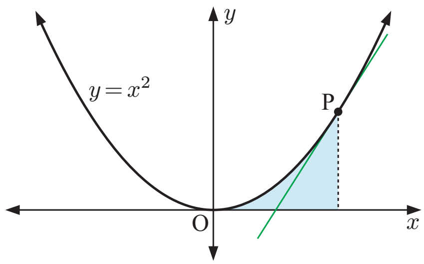
13



The blue and red shaded areas are equal. Find k, giving your answer in the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$.

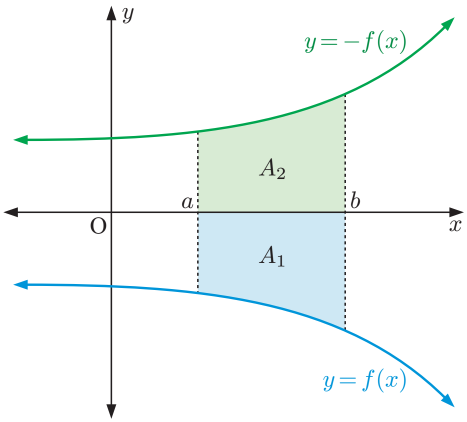
© HAESE MATHEMATICS - SAMPLE

14 Show that, for any point P on the curve $y = x^2$, the tangent at P divides the shaded area in the ratio $1:3$.



THE AREA BETWEEN $y = f(x)$ AND THE x-AXIS WHEN $f(x) \leqslant 0$

In the graph alongside, the function $y = f(x)$ lies *below* the x-axis for $a \leqslant x \leqslant b$. The shaded area A_1 does *not* equal $\int_a^b f(x)\,dx$, as this is only true if $f(x) \geqslant 0$ on $a \leqslant x \leqslant b$.



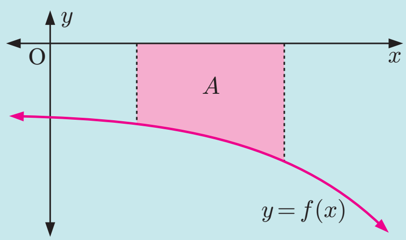
By reflecting $y = f(x)$ in the x-axis, we generate the graph of $y = -f(x)$. This graph is positive for $a \leqslant x \leqslant b$, and the area A_2 must equal A_1.

Now $A_2 = \displaystyle\int_a^b (-f(x))\,dx$.

$\therefore\ A_1 = \displaystyle\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx.$

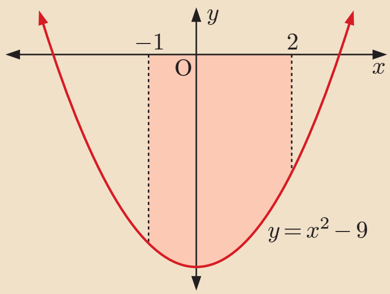
If a function $y = f(x)$ lies below the x-axis for $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is given by

$$A = -\int_a^b f(x)\,dx.$$



Example 3 — Self Tutor

Calculate the shaded area.

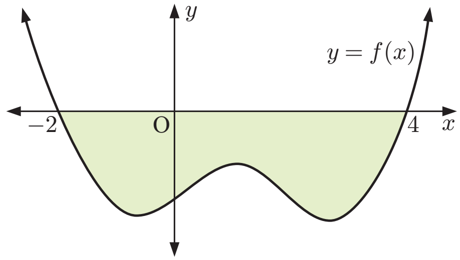


$$\begin{aligned}
\text{Area} &= -\int_{-1}^{2} (x^2 - 9)\,dx \\
&= -\left[\frac{x^3}{3} - 9x\right]_{-1}^{2} \\
&= -\left(\left(\tfrac{8}{3} - 18\right) - \left(-\tfrac{1}{3} + 9\right)\right) \\
&= -(-24) \\
&= 24 \text{ units}^2
\end{aligned}$$

© HAESE MATHEMATICS - SAMPLE

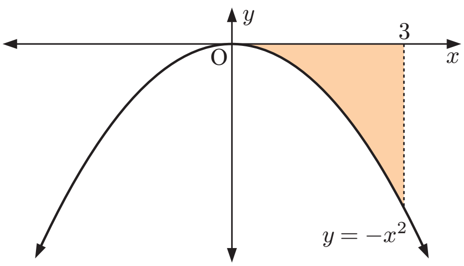
EXERCISE 17A.2

1 Write an integral which could be used to calculate the shaded area.

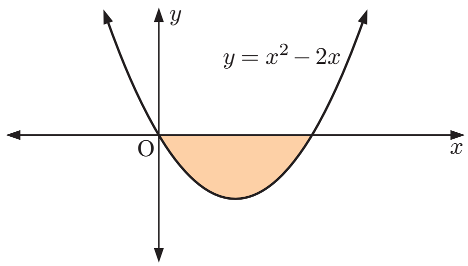


2 Find the shaded area:

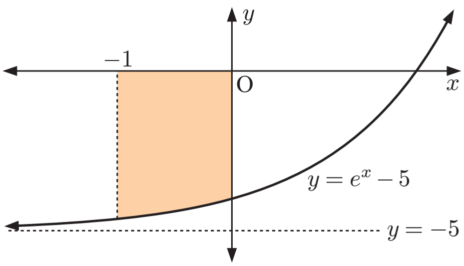
a



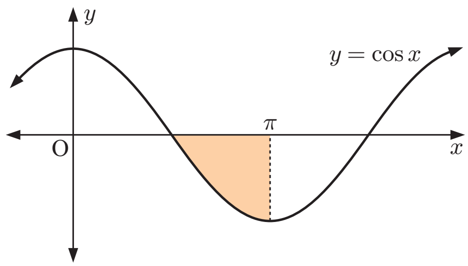
b



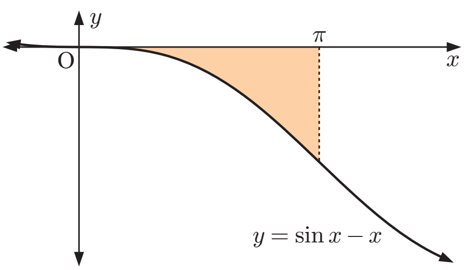
c



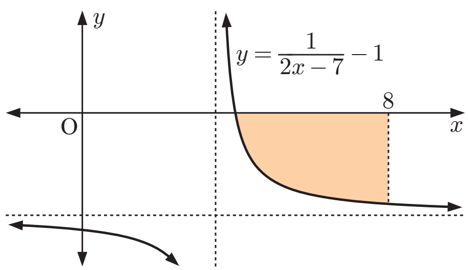
d



e



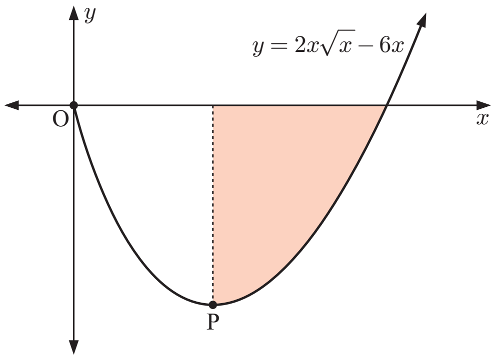
f



3 In the graph alongside, P is a minimum turning point.

a Find the coordinates of P.

b Find the shaded area.



© HAESE MATHEMATICS - SAMPLE

Example 4

Self Tutor

Find the total area of the regions bounded by $f(x) = 4x^3 - 12x^2 - 16x$ and the x-axis.

$f(x) = 4x^3 - 12x^2 - 16x$
$= 4x(x^2 - 3x - 4)$
$= 4x(x+1)(x-4)$

$\therefore$ $y = f(x)$ cuts the x-axis at -1, 0, and 4.

Total area

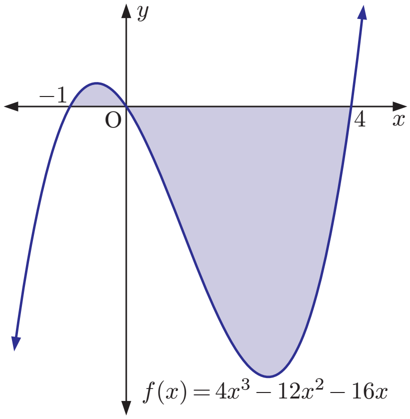
$= \int_{-1}^{0} (4x^3 - 12x^2 - 16x)\,dx - \int_{0}^{4} (4x^3 - 12x^2 - 16x)\,dx$

$= \left[x^4 - 4x^3 - 8x^2\right]_{-1}^{0} - \left[x^4 - 4x^3 - 8x^2\right]_{0}^{4}$

$= (0 - (-3)) - (-128 - 0)$

$= 3 + 128$

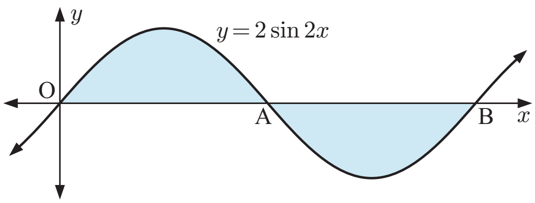
$= 131$ units^2



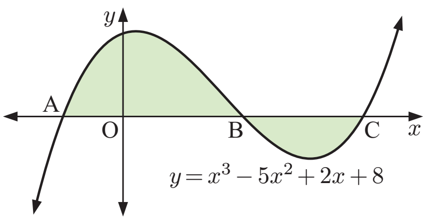
4 Consider the graph of $y = 2\sin 2x$ shown.

a Find the coordinates of A and B.

b Find the total shaded area.



5



The graph of $y = x^3 - 5x^2 + 2x + 8$ is shown alongside.

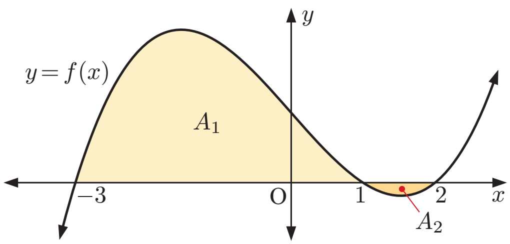
a Given that A has coordinates $(-1, 0)$, find the coordinates of B and C.

b Find the total shaded area.

6 State, in terms of the areas A_1 and A_2, what is represented by:

a $\int_{-3}^{2} f(x)\,dx$

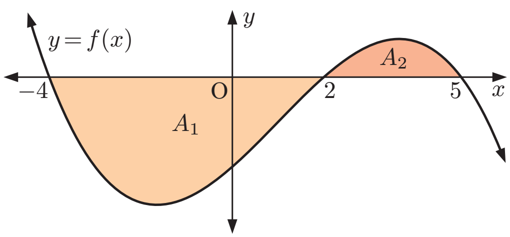
b $\int_{-3}^{1} f(x)\,dx - \int_{1}^{2} f(x)\,dx$



7 In the graph alongside, $\int_{-4}^{5} f(x)\,dx = -3$.

a Which of A_1 or A_2 is larger? Explain your answer.

b Given that $\int_{2}^{5} f(x)\,dx = 8$, find the total area of the shaded region.



© HAESE MATHEMATICS - SAMPLE

Discussion

Suppose $\int_a^b f(x)\,dx = 5$. Discuss whether each of the following statements is definitely true, possibly true, or definitely false.

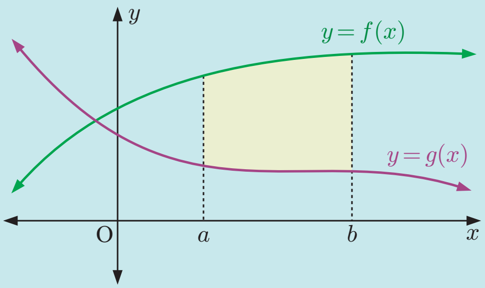
- $f(x)$ lies entirely above the x-axis over $a \leqslant x \leqslant b$.
- $f(x)$ lies entirely below the x-axis over $a \leqslant x \leqslant b$.
- The total area between $y = f(x)$ and the x-axis over $a \leqslant x \leqslant b$ is 5 units^2.
- The total area between $y = f(x)$ and the x-axis over $a \leqslant x \leqslant b$ is greater than 5 units^2.
- The total area between $y = f(x)$ and the x-axis over $a \leqslant x \leqslant b$ is less than 5 units^2.
- The area enclosed by $y = f(x)$ *above* the x-axis is 5 units^2 greater than the area *below* the x-axis over $a \leqslant x \leqslant b$.

B THE AREA BETWEEN TWO FUNCTIONS

Consider two functions $f(x)$ and $g(x)$ where $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$.

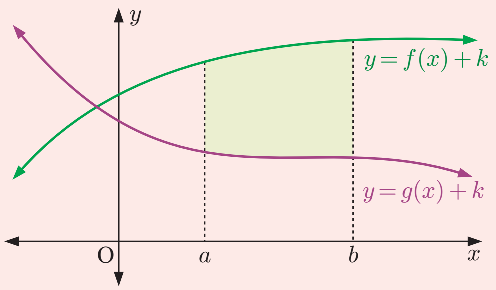
The area between the two functions on the interval $a \leqslant x \leqslant b$ is given by

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



Proof:

If necessary, we translate each curve k units upwards until they are both above the x-axis on the interval $a \leqslant x \leqslant b$. This has no effect on the area between the functions.



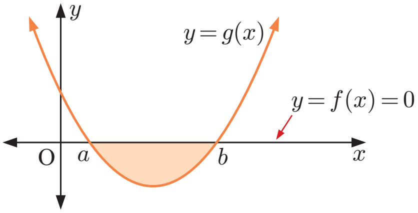
The area of the shaded region

$$= \int_a^b [f(x) + k]\,dx - \int_a^b [g(x) + k]\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

Notice that if $y = g(x)$ is negative on the interval between two x-intercepts a and b, we can let $f(x) = 0$ and hence derive the formula we saw in the last Section:

$$\text{Area} = -\int_a^b g(x)\,dx.$$

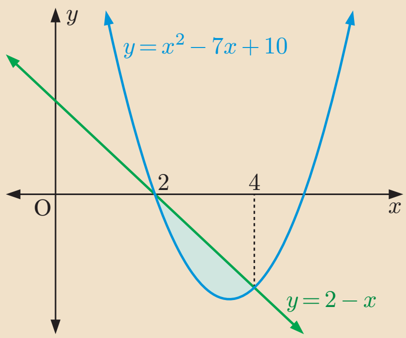


© HAESE MATHEMATICS - SAMPLE

Example 5

Self Tutor

Find the shaded area:



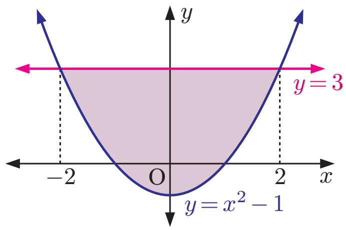
Since $2 - x \geqslant x^2 - 7x + 10$ on the interval $2 \leqslant x \leqslant 4$,

$$\begin{aligned}\text{area} &= \int_2^4 \big((2-x) - (x^2 - 7x + 10)\big)\,dx \\ &= \int_2^4 (-x^2 + 6x - 8)\,dx \\ &= \left[-\frac{x^3}{3} + 3x^2 - 8x\right]_2^4 \\ &= \left(-\tfrac{64}{3} + 48 - 32\right) - \left(-\tfrac{8}{3} + 12 - 16\right) \\ &= 1\tfrac{1}{3}\ \text{units}^2\end{aligned}$$

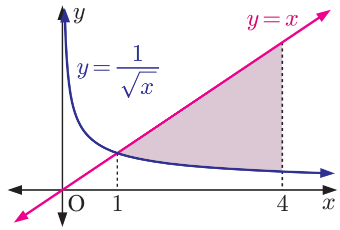
EXERCISE 17B

1 Find the shaded area:

a

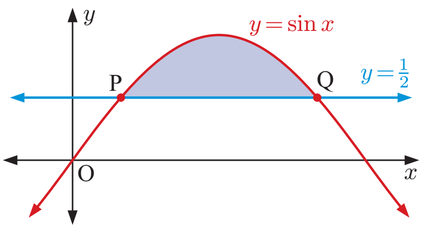


b



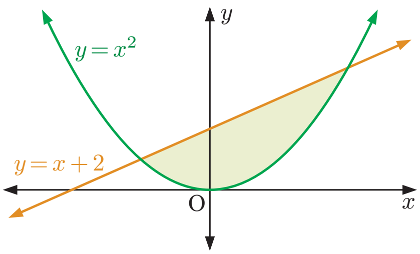
2 **a** Find the coordinates of P and Q.

b Hence find the shaded area.

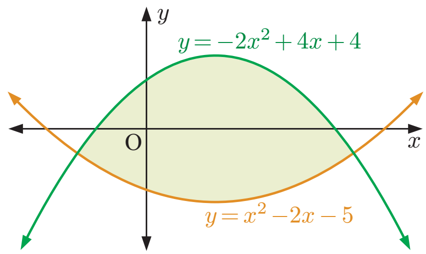


3 Find the shaded area:

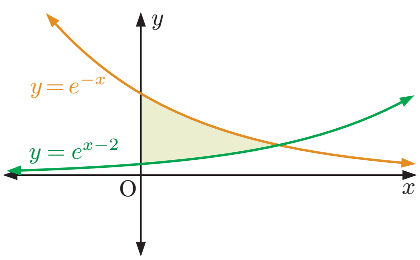
a



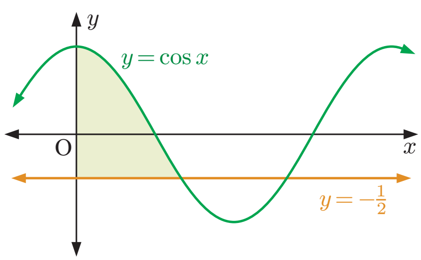
b



c



d



© HAESE MATHEMATICS - SAMPLE

4 **a** Sketch the graphs of $y = x^2$ and $y = 3x$ on the same set of axes.
b Find the coordinates of the points of intersection.
c Find the area of the region enclosed by the functions.

5 **a** Sketch the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$ on the same set of axes.
b Find the coordinates of the points of intersection.
c Find the area of the region enclosed by the curves.

Example 6

Self Tutor

Find the total area of the regions enclosed by $f(x) = x^3 - 4x$ and $g(x) = x^2 + 2x$.

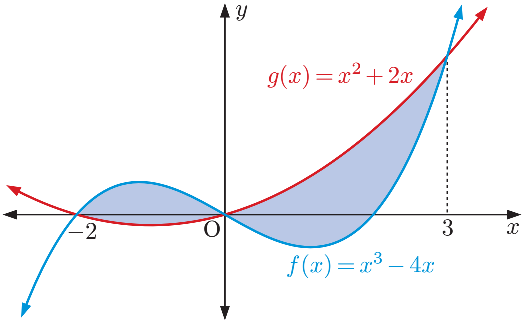
The graphs meet when $x^3 - 4x = x^2 + 2x$

$\therefore\ x^3 - x^2 - 6x = 0$

$\therefore\ x(x^2 - x - 6) = 0$

$\therefore\ x(x+2)(x-3) = 0$

$\therefore$ the graphs meet when $x = -2$, 0, and 3.



Total area

$$= \int_{-2}^{0} \left((x^3 - 4x) - (x^2 + 2x)\right) dx + \int_{0}^{3} \left((x^2 + 2x) - (x^3 - 4x)\right) dx$$

$$= \int_{-2}^{0} (x^3 - x^2 - 6x)\, dx + \int_{0}^{3} (-x^3 + x^2 + 6x)\, dx$$

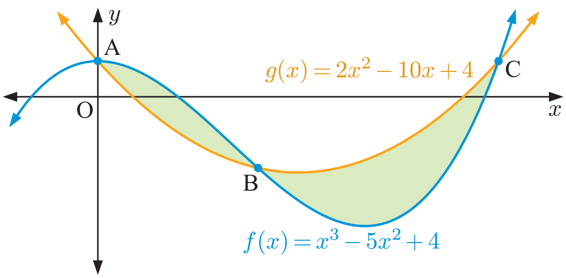
$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0} + \left[-\frac{x^4}{4} + \frac{x^3}{3} + 3x^2\right]_{0}^{3}$$

$$= \left(0 - \left(4 + \tfrac{8}{3} - 12\right)\right) + \left(-\tfrac{81}{4} + 9 + 27 - 0\right)$$

$$= \tfrac{16}{3} + \tfrac{63}{4}$$

$$= 21\tfrac{1}{12} \text{ units}^2$$

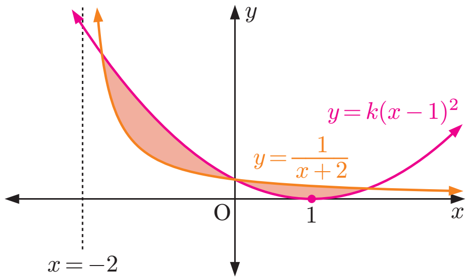
6 The graphs of $f(x) = x^3 - 5x^2 + 4$ and $g(x) = 2x^2 - 10x + 4$ are shown alongside.
a Find the coordinates of A, B, and C.
b Find the total area enclosed by the curves.



7 Find the total area enclosed by the graphs of $f(x) = x^3 - 3x + 2$ and $g(x) = 2x^2 + 2x - 4$.

© HAESE MATHEMATICS - SAMPLE

8



The graphs of $y = \frac{1}{x+2}$ and $y = k(x-1)^2$ meet at the y-axis as shown.

a Find the value of k.

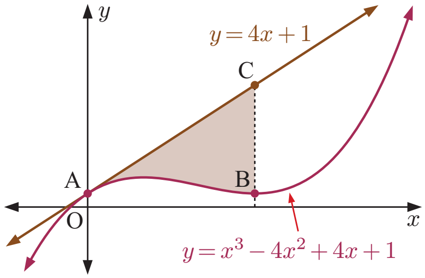
b Find the area of the shaded region.

9 The graphs of $y = 4x + 1$ and $y = x^3 - 4x^2 + 4x + 1$ are shown alongside. The graphs meet at A. B is a minimum turning point, and the vertical line through B meets $y = 4x + 1$ at C.

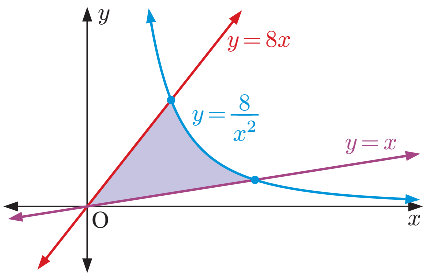
a Show that the line AC is a tangent to the curve $y = x^3 - 4x^2 + 4x + 1$ at A.

b Find the coordinates of B.

c Find the area of the shaded region.



10



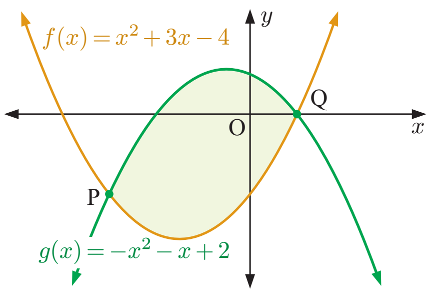
Find the shaded area.

11 Consider the graphs of $f(x) = x^2 + 3x - 4$ and $g(x) = -x^2 - x + 2$.

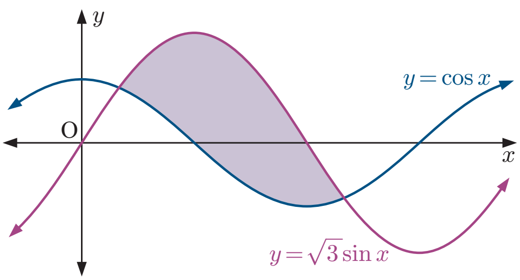
a Find the coordinates of P and Q.

b Find the area of the shaded region.

c Show that the straight line PQ divides the shaded region in half.



12

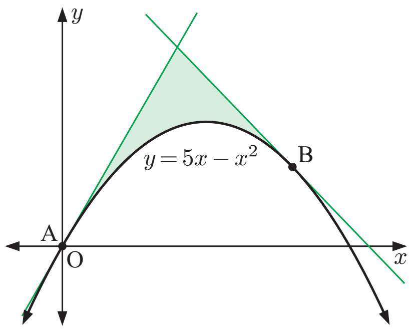


a Show that the shaded region has area 4 units^2.

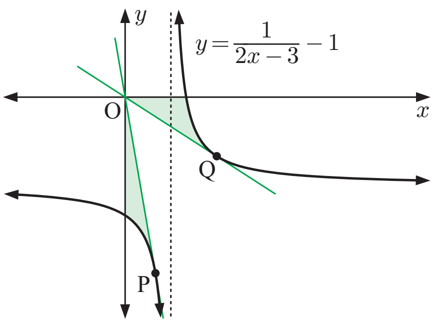
b What percentage of the shaded region lies above the x-axis?

© HAESE MATHEMATICS - SAMPLE

13 The graph alongside shows the tangents to the curve $y = 5x - x^2$ at A(0, 0) and B(4, 4).
Find the area of the shaded region.



14



The tangents to $y = \dfrac{1}{2x-3} - 1$ at P and Q pass through the origin. Find the total area of the shaded regions.

C KINEMATICS

DISPLACEMENT AND DISTANCE

In **Chapter 15**, we saw that the velocity $v(t)$ of an object is the derivative of its displacement $s(t)$, so $v(t) = s'(t)$.

The change in displacement of an object from time $t = t_1$ to $t = t_2$ is therefore

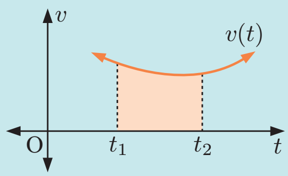
$$\begin{aligned} s(t_2) - s(t_1) &= \int_{t_1}^{t_2} s'(t)\, dt \qquad \{\text{Fundamental Theorem of Calculus}\} \\ &= \int_{t_1}^{t_2} v(t)\, dt \end{aligned}$$

If $v(t)$ maintains the same sign in the interval $t_1 \leqslant t \leqslant t_2$, then the object does not change direction.

Suppose an object travels with velocity function $v(t)$.

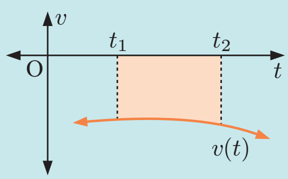
- If $v(t) \geqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$,

  distance travelled $= \displaystyle\int_{t_1}^{t_2} \boldsymbol{v(t)\, dt}$



- If $v(t) \leqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$,

  distance travelled $= -\displaystyle\int_{t_1}^{t_2} \boldsymbol{v(t)\, dt}$



© HAESE MATHEMATICS - SAMPLE