Chapter 14

Applications of differentiation

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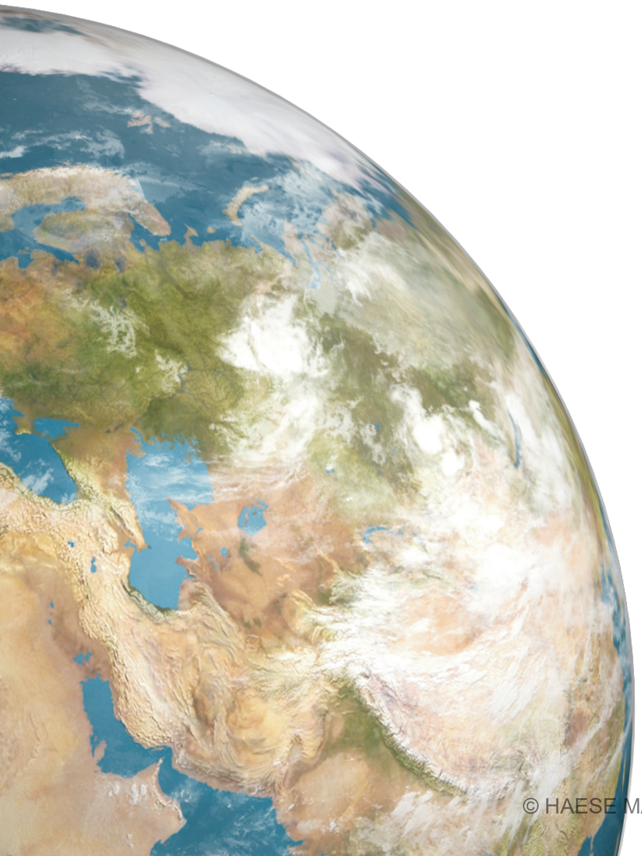


OPENING PROBLEM

On the Indonesian coast, the depth of water t hours after midnight is given by $D = 9.3 + 6.8\cos(0.507t)$ metres.

Things to think about:

a What is the derivative function $\frac{dD}{dt}$ and what does it tell us?

b What is the depth of water at 8 am?

c Is the tide rising or falling at 8 am? Explain your answer.

d At what time(s) is the tide highest on this day? What is the maximum depth of water?

We have already seen that if $y = f(x)$ then $f'(x)$ or $\frac{dy}{dx}$ gives the gradient of the tangent to $y = f(x)$ for any value of x.

In this Chapter we consider some real-world applications of differential calculus, using derivatives to tell us how one variable changes relative to another.

A RATES OF CHANGE

There are countless examples in the real world where quantities vary with time, or with respect to some other variable.

For example:

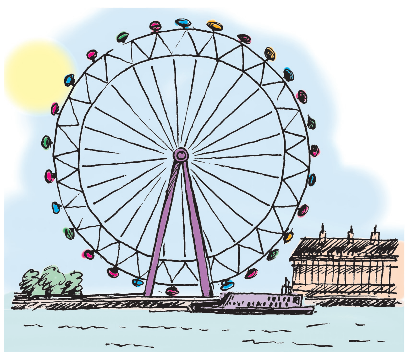
- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

$\boldsymbol{\frac{dy}{dx}}$ gives the **rate of change in $\boldsymbol{y}$ with respect to $\boldsymbol{x}$**.

We can therefore use the derivative of a function to tell us the **rate** at which something is happening.

For example:

- $\frac{dH}{dt}$ or $H'(t)$ could be the instantaneous rate of ascent of a person in a Ferris wheel.
  It might have units metres per second or m s^{-1}.
- $\frac{dC}{dt}$ or $C'(t)$ could be a person's instantaneous rate of change in lung capacity.
  It might have units litres per second or L s^{-1}.



Example 1 **Self Tutor**

According to a psychologist, the ability of a child to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a child is:

i 9 years old **ii** 16 years old.

b Show that $\dfrac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Show that $\dfrac{d^2A}{dt^2} < 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

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a $$A = \tfrac{1}{3}\sqrt{t} = \tfrac{1}{3}t^{\frac{1}{2}}$$

$$\therefore \frac{dA}{dt} = \tfrac{1}{6}t^{-\frac{1}{2}} = \frac{1}{6\sqrt{t}}$$

i When $t = 9$, $\dfrac{dA}{dt} = \frac{1}{18}$

$\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old child.

ii When $t = 16$, $\dfrac{dA}{dt} = \frac{1}{24}$

$\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old child.

b Since $\sqrt{t}$ is never negative, $\dfrac{1}{6\sqrt{t}}$ is never negative

$\therefore \dfrac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $$\frac{dA}{dt} = \tfrac{1}{6}t^{-\frac{1}{2}}$$

$$\therefore \frac{d^2A}{dt^2} = -\tfrac{1}{12}t^{-\frac{3}{2}} = -\frac{1}{12t\sqrt{t}}$$

$$\therefore \frac{d^2A}{dt^2} < 0 \quad \text{for all } 5 \leqslant t \leqslant 18.$$

This means that while the ability to understand spatial concepts increases with age, the rate of increase slows down with age.

You are encouraged to use technology to graph each function you need to consider. This is often useful in interpreting results.

GRAPHING PACKAGE

EXERCISE 14A

1 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

a What is the current annual profit?

b Find $\dfrac{dP}{dt}$ and state its units.

c Find $\dfrac{dP}{dt}$ when $t = 8$. Explain what this value means.

2 In a hot, dry summer, water is evaporating from a desert oasis. The volume of water remaining after t days is $V = 2(50-t)^2$ m^3. Find:

a the average rate at which the water evaporates in the first 5 days

b the instantaneous rate at which the water is evaporating at $t = 5$ days.

3 The quantity of a chemical in human skin which is responsible for its "elasticity" is given by $Q(t) = 100 - 10\sqrt{t}$ where t is the age of a person in years.

a Find $Q(t)$ when:

i $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.

b At what rate is the quantity of the chemical changing when the person is aged:

i 25 years **ii** 50 years?

c Show that the quantity of the chemical is decreasing for all $t > 0$.

4 The height of *pinus sylvestris* is given by

$H = 35 - \dfrac{172.5}{t+5}$ metres, where t is the number of years after the tree was planted from an established seedling.

a How high was the tree when it was planted?

b Find the height of the tree after:

i 4 years **ii** 8 years **iii** 12 years.

c Find the rate at which the tree was growing after 0, 5, and 10 years.

d Show that $\dfrac{dH}{dt} > 0$ for all $t \geqslant 0$. Explain the significance of this result.

Example 2 **Self Tutor**

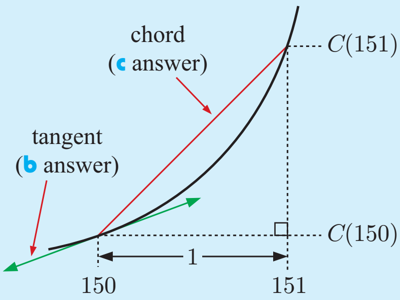
The cost in dollars of producing x items in a factory each day is given by $C(x) = 9500 + 12x + 8x^{0.8}$.

a Find $C'(x)$, which is called the marginal cost function.

b Find the marginal cost when 150 items are produced. Interpret this result.

c Find $C(151) - C(150)$. Compare this with the answer in **b**.

a The marginal cost function is $C'(x) = 12 + 6.4x^{-0.2}$ dollars per item.



b $C'(150) \approx \$14.35$

This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day.

It gives an estimate of the cost of making the 151st item each day.

c $C(151) - C(150) \approx \$11\,754.87 - \$11\,740.52$

$\approx \$14.35$

This is the actual cost of making the 151st item each day, so the answer in **b** gives a very good estimate.

5 Seablue make denim jeans. The cost model for making x pairs per day is

$$C(x) = 7800 + 6x + 12x^{0.7} \text{ dollars.}$$

a Find the marginal cost function $C'(x)$.

b Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. Discuss your answer.

6 The total cost of running a train from Paris to Marseille is given by $C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v}$ euros where v is the average speed of the train in km h^{-1}.

a Find the total cost of the journey if the average speed is:

i $50\ \text{km h}^{-1}$ **ii** $100\ \text{km h}^{-1}$.

b Find the rate of change in the cost of running the train for the average speed:

i $30\ \text{km h}^{-1}$ **ii** $90\ \text{km h}^{-1}$.

c At what speed will the cost be a minimum?

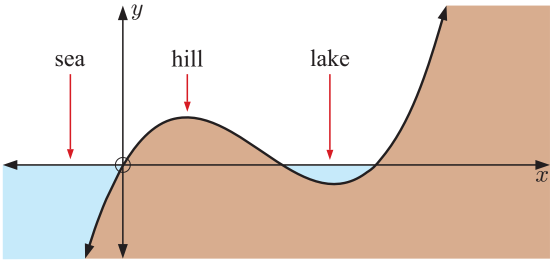
7 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ litres where $0 \leqslant t \leqslant 80$.

a Find $\frac{dV}{dt}$, and draw the graph of $\frac{dV}{dt}$ against t.

b At what time was the outflow fastest?

c Show that $\frac{d^2V}{dt^2}$ is always constant and positive. Interpret this result.

8 Alongside is a land and sea profile where the x-axis is sea level.
The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level at distance x km from the shore line.



a Find where the lake is located relative to the shore line of the sea.

b Find $\frac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ km and when $x = 1\frac{1}{2}$ km.

c Find the deepest point of the lake, and the depth at this point.

9 A radioactive substance decays according to the formula $W = 20e^{-kt}$ grams where t is the time in hours.

a Find k given that the weight is 10 grams after 50 hours.

b Find the weight of radioactive substance present:

i initially **ii** after 24 hours **iii** after 1 week.

c How long will it take for the weight to reach 1 gram?

d Find the rate of radioactive decay after:

i 100 hours **ii** 1000 hours.

e Show that $\frac{dW}{dt} = bW$ for some constant b.

10 The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-kt}$ °C where k is a positive constant and t is the time in minutes.

a Find k if the temperature of the liquid is 20°C after 15 minutes.

b What was the temperature of the liquid when it was first placed in the refrigerator?

c Show that $\dfrac{dT}{dt} = c(T - 5)$ for some constant c.

d At what rate is the temperature changing:

i initially **ii** after 10 minutes **iii** after 20 minutes?

11 The height of a shrub t years after it was planted is given by $H(t) = 20\ln(3t + 2) + 30$ cm, $t \geqslant 0$.

a How high was the shrub when it was planted?

b How long will it take for the shrub to reach a height of 1 m?

c At what rate is the shrub's height changing:

i 3 years after being planted **ii** 10 years after being planted?

12 In the conversion of sugar solution to alcohol, the amount of alcohol produced t hours after the reaction commenced is given by $A = s(1 - e^{-kt})$ litres, where s is the original sugar concentration (%), $t \geqslant 0$.

a Find A when $t = 0$.

b Suppose $s = 10$, and $A = 5$ after 3 hours.

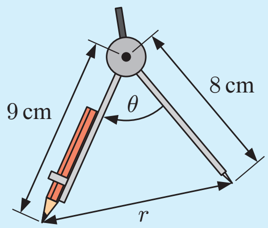
i Find k. **ii** Find the speed of the reaction after 5 hours.

Example 3 **Self Tutor**

Cathy is using a compass to draw a circle. The arm with the needle is 8 cm long, and the arm with the pencil is 9 cm in total. The angle between the arms is θ.

a Write the radius of the circle to be drawn in terms of θ.

b Hence find the rate of change in r with respect to θ when $\theta = 60°$.



a Using the cosine rule, $r^2 = 9^2 + 8^2 - 2 \times 9 \times 8 \times \cos\theta$

$\therefore \ r = \sqrt{145 - 144\cos\theta}$ {since $r > 0$}

b $r = (145 - 144\cos\theta)^{\frac{1}{2}}$

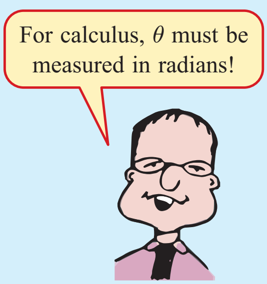
$$\therefore \ \frac{dr}{d\theta} = \tfrac{1}{2}(145 - 144\cos\theta)^{-\frac{1}{2}}(144\sin\theta)$$

$$= \frac{72\sin\theta}{\sqrt{145 - 144\cos\theta}}$$

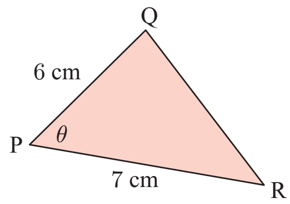
When $\theta = \frac{\pi}{3}$, $\dfrac{dr}{d\theta} = \dfrac{72\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{145 - 144\left(\frac{1}{2}\right)}}$

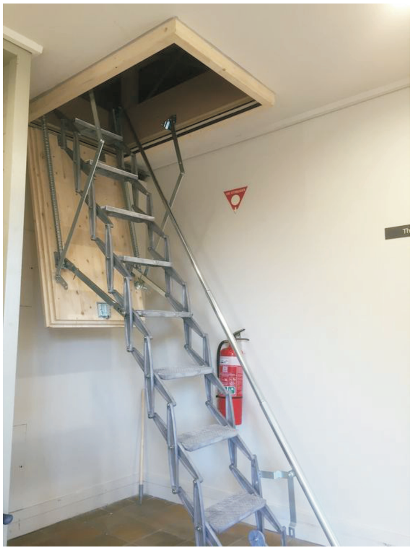
≈ 7.30 cm per radian

≈ 0.127 cm per degree

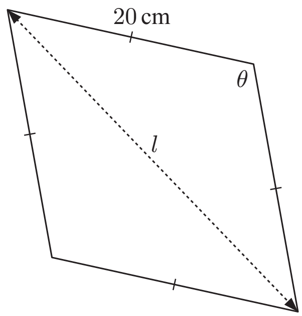


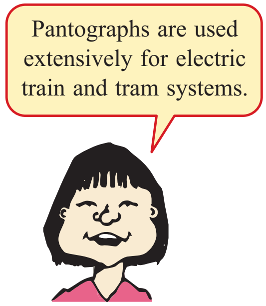
13 Find exactly the rate of change in the area of triangle PQR as θ changes, at the time when $\theta = 45°$.



14 

A set of retractable stairs is used to gain access to an attic. The frame uses a pantograph mechanism which is a set of rhombuses with variable angles to control the retraction. Each rhombus has side length 20 cm, and the angle where the arms meet is θ, as shown.





a Find a formula for the length l between the pivots, in terms of θ.

b Hence find the rate of change in l at the time when $\theta = 120°$.

15 The voltage in a circuit is given by $V(t) = 340\sin(100\pi t)$ volts where t is the time in seconds.

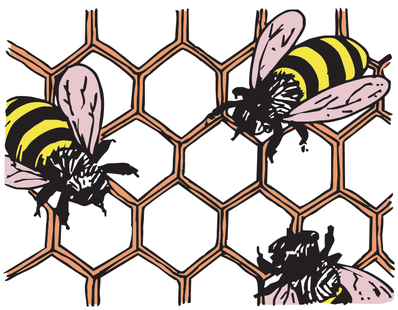
a Find the voltage in the circuit:

i initially **ii** after 0.125 seconds.

b At what rate is the voltage changing:

i when $t = 0.01$ **ii** when $V(t)$ is a maximum?

16 The number of bees in a hive after t months is modelled by $B(t) = \dfrac{3000}{1 + 0.5e^{-1.73t}}$.



a Find the initial bee population.

b Find the percentage increase in the population after 1 month.

c Is there a limit to the population size? If so, what is it?

d Find $B'(t)$, and use it to explain why the population is increasing over time.

e Find the rate at which the population is increasing after 6 months.

f Sketch the graph of $B(t)$.

B OPTIMISATION

Optimisation is the process of finding the **maximum** or **minimum** value of a function. The solution is often referred to as the **optimal solution**.

We can find optimal solutions in several ways:

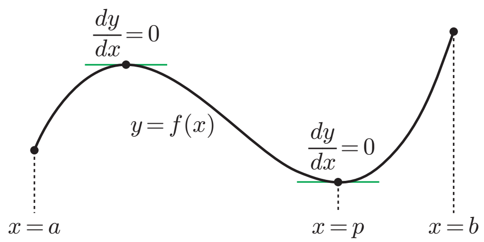
- using technology to graph the function and search for the maximum or minimum value
- using analytical methods such as the formula $x = -\frac{b}{2a}$ for the vertex of a parabola
- using differential calculus to locate the turning points of a function.

These last two methods are useful especially when exact solutions are required.

You should always be aware that:

> The maximum or minimum value does not always occur when the first derivative is zero.
>
> It is essential to also examine the values of the function at the end point(s) of the interval under consideration for global maxima and minima.

For example:



The maximum value of y occurs at the end point $x = b$.

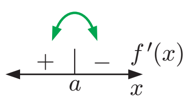
The minimum value of y occurs at the local minimum $x = p$.

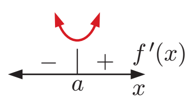
TESTING FOR LOCAL MAXIMA AND MINIMA

If we find a value $x = a$ such that $f'(a) = 0$, there are several tests we can use to see whether we have a local maximum or a local minimum at this point.

SIGN DIAGRAM TEST

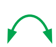
If, near to $x = a$, the sign diagram of $f'(x)$ is:

- 

we have a **local maximum**
- 

we have a **local minimum**.

SECOND DERIVATIVE TEST

- If $f''(a) < 0$ we have  shape, which indicates a **local maximum**.
- If $f''(a) > 0$ we have  shape, which indicates a **local minimum**.

OPTIMISATION PROBLEM SOLVING METHOD

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example x. You should write down what domain restrictions there are on x.

Step 3: Find the **first derivative** and find the value(s) of x which make the first derivative **zero**.

Step 4: For each stationary point, use the **sign diagram test** or **second derivative test** to determine whether you have a local maximum or local minimum.

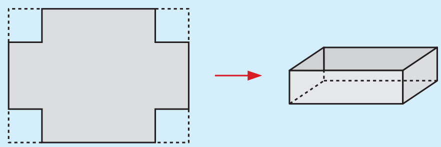
Step 5: Identify the optimal solution, also considering end points where appropriate.

Step 6: Write your answer in a sentence, making sure you specifically answer the question.

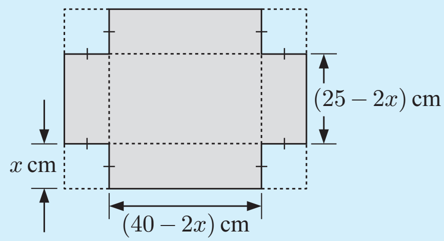
Example 4 Self Tutor

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



Step 1: Let x cm be the side lengths of the squares that are cut out.



Step 2: Volume = length × width × depth

$$= (40-2x)(25-2x)x$$
$$= (1000-80x-50x+4x^2)x$$
$$= 1000x - 130x^2 + 4x^3 \text{ cm}^3$$

Since the side lengths must be positive, $x > 0$ and $25 - 2x > 0$.

$\therefore \quad 0 < x < 12.5$

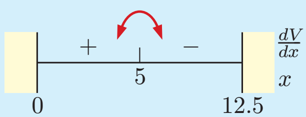
Step 3: $$\frac{dV}{dx} = 12x^2 - 260x + 1000$$
$$= 4(3x^2 - 65x + 250)$$
$$= 4(3x-50)(x-5)$$

$\therefore \quad \dfrac{dV}{dx} = 0$ when $x = \frac{50}{3} = 16\frac{2}{3}$ or $x = 5$

$\therefore \quad x = 5$ as $0 < x < 12.5$

DEMO

Step 4: $\dfrac{dV}{dx}$ has sign diagram:

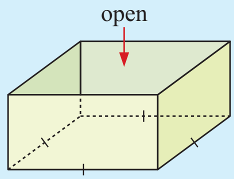


Step 5: There is a local maximum when $x = 5$. This is the global maximum for the given domain.

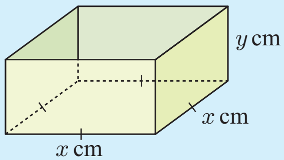
Step 6: The maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

Example 5 — Self Tutor

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.



Step 1:



Let the base lengths be x cm and the depth be y cm.

The volume $V = \text{length} \times \text{width} \times \text{depth}$

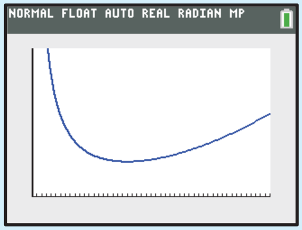
$\therefore\ V = x^2y$

$\therefore\ 4000 = x^2y \quad .... \ (1) \quad \{1 \text{ litre} \equiv 1000 \text{ cm}^3\}$

Step 2: The total surface area

$$A = \text{area of base} + 4(\text{area of one side})$$
$$= x^2 + 4xy$$
$$= x^2 + 4x\left(\frac{4000}{x^2}\right) \qquad \{\text{using (1)}\}$$

$\therefore\ A(x) = x^2 + 16\,000x^{-1}$ where $x > 0$



Step 3: $\therefore\ A'(x) = 2x - 16\,000x^{-2}$

$\therefore\ A'(x) = 0$ when $2x = \dfrac{16\,000}{x^2}$

$\therefore\ 2x^3 = 16\,000$

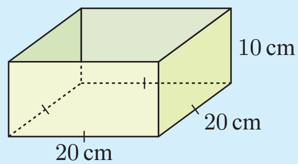
$\therefore\ x = \sqrt[3]{8000} = 20$

Step 4: $A''(x) = 2 + 32\,000x^{-3}$

$\therefore\ A''(20) = 2 + \dfrac{32\,000}{20^3} = 6$

Since $A''(20) > 0$, there is a local minimum at $x = 20$.

Step 5: The minimum material is used to make the container when $x = 20$ and $y = \dfrac{4000}{20^2} = 10$.



Step 6: The most economical shape has a square base $20 \text{ cm} \times 20 \text{ cm}$, and height 10 cm.

Use **calculus techniques** to answer the following problems.

In cases where finding the zeros of the derivatives is difficult you may use the **graphing package** to help you.

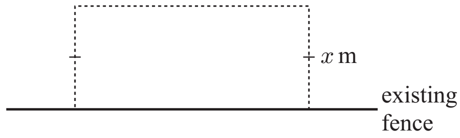
GRAPHING PACKAGE

EXERCISE 14B

1. When a manufacturer makes x items per day, the profit function is $P(x) = -0.022x^2 + 11x - 720$ pounds. Find the production level that will maximise profits.

2. The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ pounds, and for this production level each blanket may be sold for $\left(23 - \frac{1}{2}x\right)$ pounds.
   How many blankets should be produced per day to maximise the total profit?

3 60 metres of fencing is used to build a rectangular enclosure along an existing fence. Suppose the sides adjacent to the existing fence are x m long.



a Show that the area A of the enclosure is given by $A(x) = x(60 - 2x)$ m^2.

b Find the dimensions which maximise the area of the enclosure.

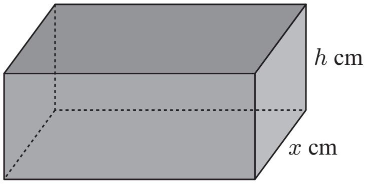
4 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides, as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.

a If the sides adjacent to the existing fence have length x m, show that the required length of wire netting to be purchased is $L = 2x + \frac{100}{x}$.

b Find the minimum value of L and the corresponding value of x when this occurs.

c Sketch the optimal situation, showing all dimensions.

5 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of a box has dimensions in the ratio $2 : 1$.



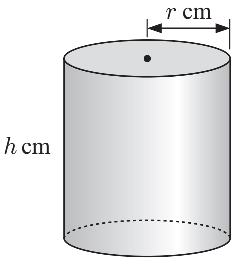
a Show that $x^2h = 100$.

b Show that the inner surface area of the box is given by $A(x) = 4x^2 + \frac{600}{x}$ cm^2.

c Find the minimum inner surface area of the box and the corresponding value of x.

d Sketch the optimal box shape, showing all dimensions.

6 Brenda is designing a cylindrical tin can for a canned fruit company. The cans must have capacity 1 litre, and they must use as little metal as possible.

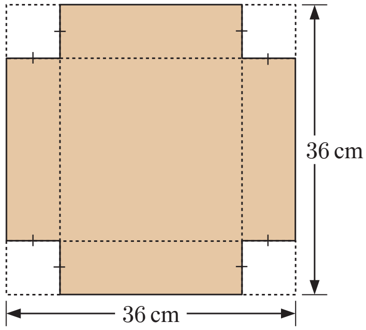


a Explain why the height h is given by $h = \frac{1000}{\pi r^2}$ cm.

b Show that the total surface area A is given by $A = 2\pi r^2 + \frac{2000}{r}$ cm^2.

c Find the dimensions of the can which make A as small as possible.

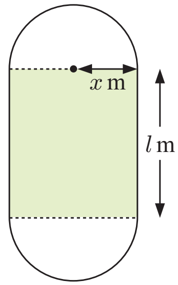
7 Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are x cm by x cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.



a Show that the volume of the container is given by $V(x) = x(36 - 2x)^2$ cm^3.

b What sized squares should be cut out to produce the container of greatest capacity?

8 An athletics track has two "straights" of length l m, and two semi-circular ends of radius x m. The perimeter of the track is 400 m.



a Show that $l = 200 - \pi x$ and write down the possible values that x may have.

b What values of l and x maximise the shaded rectangle inside the track? What is this maximum area?

9 A 60 cm length of wire is bent into a rectangle with length x cm and width y cm.

a Write an expression for y in terms of x.

b Write an expression for the area $A(x)$ of the rectangle enclosed by the wire.

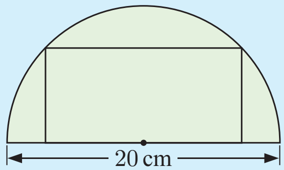
c Find $A'(x)$.

d Hence determine the value of x which maximises the area. What are the dimensions of the rectangle in this case?

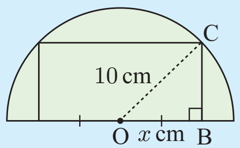
Example 6 Self Tutor

Infinitely many rectangles can be inscribed in a semi-circle of diameter 20 cm.

Find the shape of the largest rectangle which can be inscribed.



Step 1: Let OB $= x$ cm, $0 < x < 10$



In $\triangle$OBC, $\text{BC}^2 + x^2 = 10^2$ {Pythagoras}

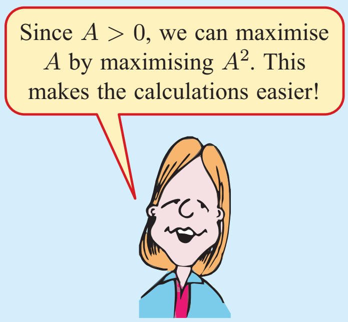
$\therefore$ BC $= \sqrt{100 - x^2}$ {as BC > 0}

Step 2: The rectangle has area $A = \text{length} \times \text{width}$

$\therefore A = 2x\sqrt{100 - x^2}$

$\therefore A^2 = 4x^2(100 - x^2)$

$= 400x^2 - 4x^4$

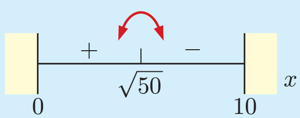


Step 3: $\dfrac{d}{dx}(A^2) = 800x - 16x^3$

$= 16x(50 - x^2)$

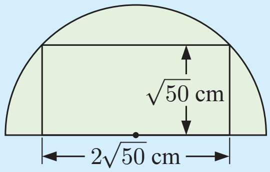
So, $\dfrac{d}{dx}(A^2) = 0$ when $x = 0$ or $\pm\sqrt{50}$.

Step 4: $\dfrac{d}{dx}(A^2)$ has sign diagram:



Step 5: The area is maximised when $x = \sqrt{50}$ and $\text{BC} = \sqrt{100 - 50}$
$= \sqrt{50}$ cm

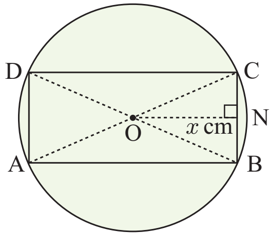
Step 6: The largest rectangle which can be inscribed is $2\sqrt{50}$ cm long and $\sqrt{50}$ cm wide.



10 Infinitely many rectangles can be inscribed in a circle of diameter 10 cm. In the diagram alongside, suppose $\text{ON} = x$ cm.

a Find the area of ABCD in terms of x only.

b Find the dimensions of ABCD which maximises its area.

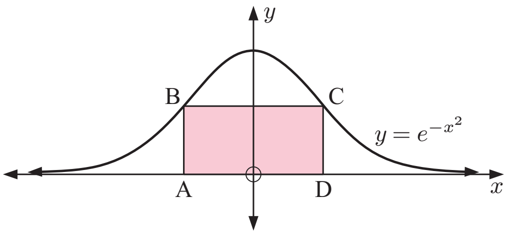


11 A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle is given by $C(x) = 4\ln x + \left(\frac{30-x}{10}\right)^2$ pounds with a minimum production capacity of 10 kettles per day.

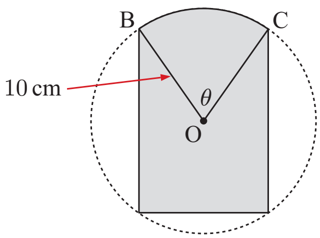
How many kettles should be manufactured to keep the cost per kettle to a minimum?

12 Infinitely many rectangles which sit on the x-axis can be inscribed under the curve $y = e^{-x^2}$.

Determine the coordinates of C such that rectangle ABCD has maximum area.



13



A circular piece of tin-plate with radius 10 cm has 3 segments removed as illustrated. The angle θ is measured in radians.

a Show that the remaining area is given by $A = 50(\theta + 3\sin\theta)$ cm^2.

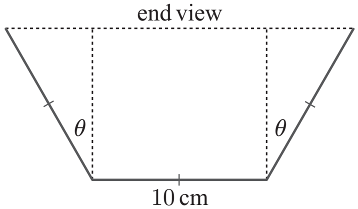
b Find θ such that the area A is a maximum, and find the area A in this case.

14 A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice as shown.

a Deduce that the cross-sectional area of the gutter is given by $A = 100\cos\theta(1 + \sin\theta)$ cm^2.

b Show that $\frac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or -1.

c For what value of θ does the gutter have maximum carrying capacity? Find the cross-sectional area for this value of θ.

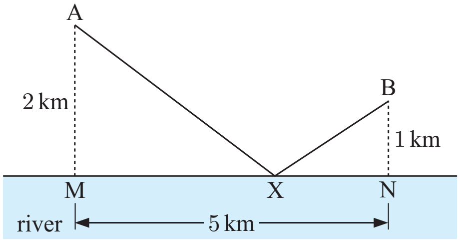


15 When a new anaesthetic is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection.

a Find $E'(t)$.

b At what time is the anaesthetic most effective?

16



A pumphouse is to be placed at some point X along a river.
Two pipelines will then connect the pumphouse to homesteads A and B.
How far should point X be from M so that the total length of pipeline is minimised?

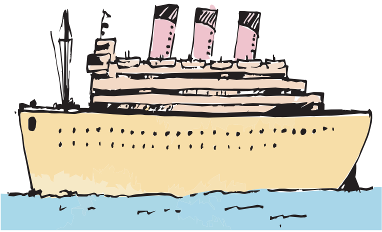
17 A small population of wasps is observed. After t weeks the population is modelled by

$P(t) = \dfrac{50\,000}{1 + 1000e^{-0.5t}}$ wasps, where $0 \leqslant t \leqslant 25$.

Find when the wasp population is growing fastest.

Hint: You need to maximise $P'(t)$.

18 At 1:00 pm ship A leaves port P. It sails in the direction 30° east of north at 12 km h^{-1}. At the same time, ship B is 100 km due east of P, and is sailing at 8 km h^{-1} towards P.

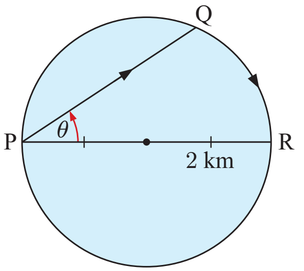


a Show that the distance between the two ships is given by $D(t) = \sqrt{304t^2 - 2800t + 10\,000}$ km, where t is the number of hours after 1:00 pm.

b Find the minimum value of D^2 for all $t \geqslant 0$.

c At what time, to the nearest minute, are the ships closest?

19 Hieu can row a boat at 3 km h^{-1}, and can walk at 6 km h^{-1}. He is currently at point P on the shore of a lake 2 km in radius. He will row to point Q, then walk around the shore to point R which is opposite P.



a Show that $\text{PQ} = 4\cos\theta$ km.

b Show that the time taken for Hieu's journey is given by $T = \frac{4}{3}\cos\theta + \dfrac{2\theta}{3}$ hours where $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

c Find θ such that $\dfrac{dT}{d\theta} = 0$ on $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

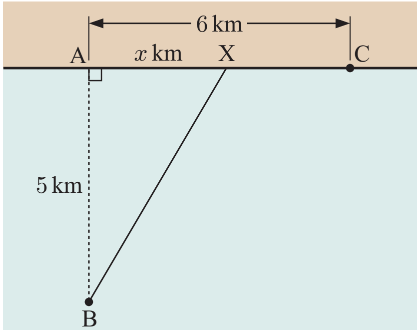
d Draw a sign diagram for $\dfrac{dT}{d\theta}$.

e What route should Hieu take to travel from P to R in:

i the longest time

ii the shortest time?

20 B is a boat 5 km out at sea from A. [AC] is a straight sandy beach, 6 km long. Peter can row the boat at 8 km h^{-1} and run along the beach at 17 km h^{-1}. Suppose Peter rows directly from B to point X on [AC] such that $AX = x$ km.



a Explain why $0 \leqslant x \leqslant 6$.

b Show that the *total time* Peter takes to row to X and then run along the beach to C, is given by

$$T = \frac{\sqrt{x^2+25}}{8} + \frac{6-x}{17} \text{ hours}, \quad 0 \leqslant x \leqslant 6.$$

c Find x such that $\frac{dT}{dx} = 0$. Explain the significance of this value.

21 A mosquito flying with position $M(x, y, z)$ is repelled by scent emitted from the origin O. At time t seconds, the coordinates of the mosquito are given by $x(t) = 3 - t^2$, $y(t) = 2 + \sqrt{t}$, and $z(t) = 2 - \sqrt{t}$, where all distance units are metres.

a Show that if the mosquito is D m from the origin at time t, then $D^2 = t^4 - 6t^2 + 2t + 17$.

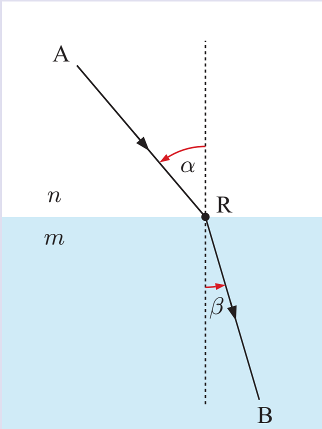
b *Hence* find the closest the mosquito came to the source of the repellent.

THEORY OF KNOWLEDGE

Snell's law states the relationship between the angles of incidence and refraction when a ray of light passes from one medium to another with different optical density. It was first discovered in 984 AD by the Persian scientist **Ibn Sahl**, who was studying the shape of lenses. However, it is named after **Willebrord Snellius**, who rediscovered it during the Renaissance. The law was published by **René Descartes** in his *Discourse on the Method* published in 1637.

Willebrord Snellius

In the figure alongside, a ray passes from A to B via point R. We suppose the refractive indices of the two media are n and m, the angle of incidence is α, and the angle of refraction is β.



Snell's law states that: $\boldsymbol{n \sin \alpha = m \sin \beta}$.

The law follows from Fermat's *principle of least time*, which says that a ray of light travelling between two points will take the path of least time.

1 Is optimisation a mathematical principle?

2 Is mathematics an intrinsic or natural part of other subjects?