Chapter 10

Points in space

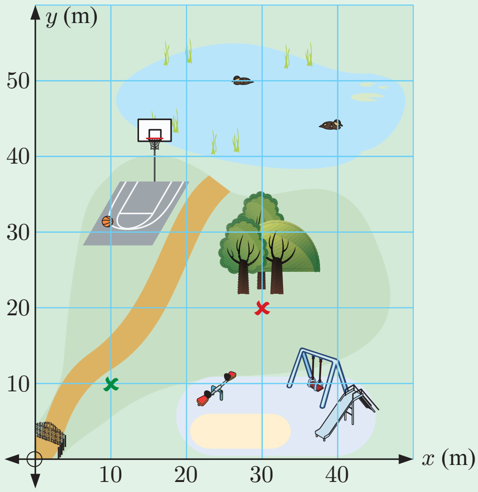
Contents:

OPENING PROBLEM

The map of a park is shown alongside.

To help us describe the location of objects in the park, we can place a 2-dimensional coordinate grid over the map, so that the origin is at the park's entrance, and each grid unit represents 10 metres.

Ayla is currently at the location marked with an ×. A bird is sitting in a tree, 10 metres above the ground. It is directly above Ayla.



Things to think about:

a How can we extend our coordinate system to describe the location of the bird?

b The bird spies a worm in the garden at ×.

i How far is the bird from the worm?

ii If the bird flies in a straight line to the worm, at what angle to the ground will it fly?

A POINTS IN SPACE

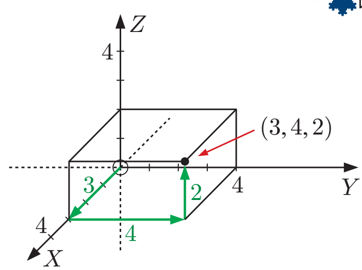
In 3-dimensional coordinate geometry, we specify an origin O, and three mutually perpendicular axes called the X-axis, the Y-axis, and the Z-axis.

3-D POINT PLOTTER

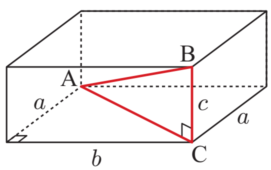
Any point in space can then be specified using an ordered triple in the form (x, y, z).

We generally suppose that the Y and Z-axes are in the plane of the page, and the X-axis is coming out of the page as shown.

The point $(3, 4, 2)$ is found by starting at the origin $\text{O}(0, 0, 0)$, moving 3 units along the X-axis, 4 units in the Y-direction, and then 2 units in the Z-direction.



We see that $(3, 4, 2)$ is located on the corner of a rectangular prism opposite O.



Now consider the rectangular prism illustrated, in which A is opposite B.

$$\text{AC}^2 = a^2 + b^2 \qquad \{\text{Pythagoras}\}$$

$$\text{and} \quad \text{AB}^2 = \text{AC}^2 + c^2 \qquad \{\text{Pythagoras}\}$$

$$\therefore \text{AB}^2 = a^2 + b^2 + c^2$$

$$\therefore \text{AB} = \sqrt{a^2 + b^2 + c^2} \qquad \{\text{AB} > 0\}$$

Suppose A is (x_1, y_1, z_1) and B is (x_2, y_2, z_2).

- The **distance** $\text{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The **midpoint** of [AB] is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

Example 1 Self Tutor

Consider $A(4, -3, 1)$ and $B(-2, 4, -1)$. Find:

a the distance AB **b** the midpoint of [AB].

a $$\begin{aligned} AB &= \sqrt{(-2-4)^2 + (4--3)^2 + (-1-1)^2} \\ &= \sqrt{(-6)^2 + 7^2 + (-2)^2} \\ &= \sqrt{36+49+4} \\ &= \sqrt{89} \text{ units} \end{aligned}$$

b The midpoint is $\left(\dfrac{4+-2}{2}, \dfrac{-3+4}{2}, \dfrac{1+-1}{2}\right)$,

which is $(1, \frac{1}{2}, 0)$.

EXERCISE 10A

1 On separate axes, plot the points:

a $(5, 0, 0)$ **b** $(0, -1, 0)$ **c** $(0, 0, 2)$ **d** $(4, 3, 0)$

e $(2, 0, -1)$ **f** $(0, 4, -3)$ **g** $(3, 1, 1)$ **h** $(2, 4, 2)$

i $(3, -2, 5)$ **j** $(-3, 3, 3)$ **k** $(-4, 3, -2)$ **l** $(-1, -4, -2)$

PRINTABLE 3-D PLOTTER PAPER

2 For each pair of points, find:

i the distance AB **ii** the midpoint of [AB].

a $A(0, 0, 0)$ and $B(6, -4, 2)$ **b** $A(4, 1, 0)$ and $B(0, 1, -2)$

c $A(1, -1, 2)$ and $B(5, -3, 0)$ **d** $A(-2, 0, 5)$ and $B(-6, 7, 3)$

e $A(-1, 5, 2)$ and $B(4, 1, -1)$ **f** $A(2, 6, -3)$ and $B(-5, 3, 2)$

3 Determine whether triangle ABC is scalene, isosceles, or equilateral:

a A is $(2, 1, -3)$, B is $(-5, 5, 3)$, and C is $(-2, 3, 6)$

b A is $(3, -1, 5)$, B is $(-1, -4, 0)$, and C is $(2, 7, -3)$

4 Suppose A is $(3, 1, -2)$, B is $(-6, 7, 13)$, and C is $(5, 9, -4)$. Show that triangle ABC is right angled.

5 Consider the points $P(1, 4, -1)$, $Q(6, -8, 7)$, and $R(-5, -2, -9)$. Let M be the midpoint of [PQ], and N be the midpoint of [QR].

a Find the coordinates of M and N.

b Show that [MN] is half the length of [PR].

6 Suppose A is $(3, 5, -2)$, and $M(-2, 8, \frac{1}{2})$ is the midpoint of [AB]. Find the coordinates of B.

7 The line segments [PQ], [QR], and [PR] have midpoints $(1, 2, \frac{3}{2})$, $(\frac{5}{2}, 3, \frac{9}{2})$, and $(-\frac{1}{2}, 6, 3)$ respectively. Find the coordinates of P, Q, and R.

8 The distance from $P(2, 4, -3)$ to $Q(k, -1, -2)$ is 7 units. Find the possible values of k.

9 Find the relationship between x, y, and z if the point $\mathrm{P}(x, y, z)$:

a is always 3 units from $\mathrm{O}(0, 0, 0)$
b is always 1 unit from $\mathrm{A}(2, 5, 4)$.

Comment on your answer in each case.

10 Illustrate and describe these sets:

a $\{(x, y, z) \mid x = 3\}$
b $\{(x, y, z) \mid y = 2, \ z = -1\}$
c $\{(x, y, z) \mid x^2 + y^2 = 4, \ z = 0\}$
d $\{(x, y, z) \mid x^2 + y^2 + z^2 = 9\}$
e $\{(x, y, z) \mid 0 \leqslant x \leqslant 4, \ 0 \leqslant y \leqslant 1, \ z = 2\}$
f $\{(x, y, z) \mid 0 \leqslant x \leqslant 3, \ 0 \leqslant y \leqslant 5, \ 0 \leqslant z \leqslant 2\}$.

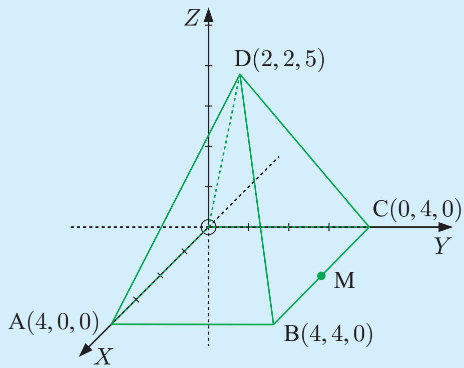
B MEASUREMENT

We can use the measurement formulae studied in **Chapter 6** to find the surface area and volume of solids in 3-dimensional space.

Example 2 Self Tutor

A square-based pyramid has base coordinates $\mathrm{O}(0, 0, 0)$, $\mathrm{A}(4, 0, 0)$, $\mathrm{B}(4, 4, 0)$, and $\mathrm{C}(0, 4, 0)$. The apex of the pyramid is $\mathrm{D}(2, 2, 5)$.

a Verify that the apex lies directly above the centre of the base.
b Find the volume of the pyramid.
c Suppose M is the midpoint of [BC].
  i Find the coordinates of M.
  ii Find the exact length of [MD].
  iii Hence find the surface area of the pyramid.



a To find the centre of the base, we locate the midpoints of the diagonals.

The midpoint of [OB] is $\left(\frac{0+4}{2}, \frac{0+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 2, 0)$.

The midpoint of [AC] is $\left(\frac{4+0}{2}, \frac{0+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 2, 0)$.

$\therefore$ the centre of the base is $(2, 2, 0)$.
$\therefore$ the apex $(2, 2, 5)$ lies directly above the centre of the base.

b
$$\begin{aligned}\text{Volume} &= \tfrac{1}{3}(\text{area of base} \times \text{height})\\ &= \tfrac{1}{3} \times 4 \times 4 \times 5\\ &= \tfrac{80}{3} \text{ units}^3\end{aligned}$$

c **i** M is $\left(\frac{4+0}{2}, \frac{4+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 4, 0)$.

ii
$$\begin{aligned}\mathrm{MD} &= \sqrt{(2-2)^2 + (2-4)^2 + (5-0)^2}\\ &= \sqrt{0^2 + (-2)^2 + 5^2}\\ &= \sqrt{29} \text{ units}\end{aligned}$$

iii Area of triangle BCD $= \frac{1}{2} \times 4 \times \sqrt{29}$

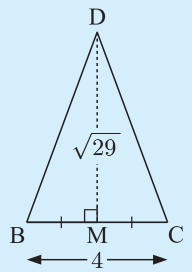
$= 2\sqrt{29}$ units^2

$\therefore$ surface area of pyramid

$=$ area of base $+$ area of 4 triangular faces

$= (4 \times 4 + 4 \times 2\sqrt{29})$ units^2

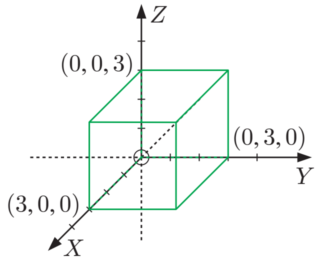
≈ 59.1 units^2



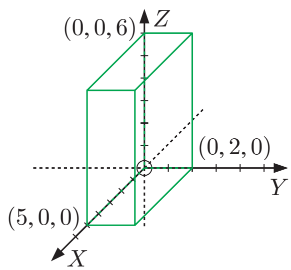
EXERCISE 10B

1 Find the volume of each rectangular prism:

a



b



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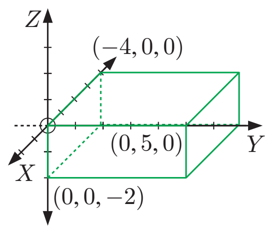


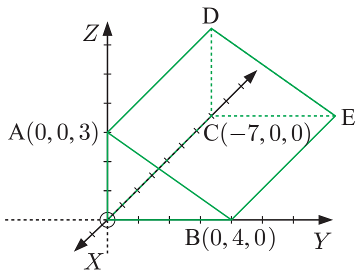
2 Consider the triangular-based prism alongside.

a State the coordinates of D and E.

b Find the volume of the prism.

c Find the length of [AB].

d Hence find the surface area of the prism.



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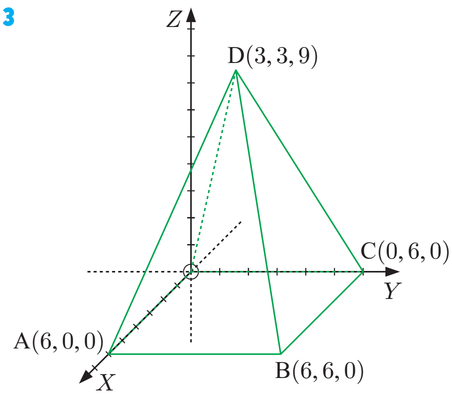


A square-based pyramid has base coordinates O(0, 0, 0), A(6, 0, 0), B(6, 6, 0), and C(0, 6, 0). The apex of the pyramid is D(3, 3, 9).

a Verify that the apex lies directly above the centre of the base.

b Find the volume of the pyramid.

c Suppose M is the midpoint of [AB].

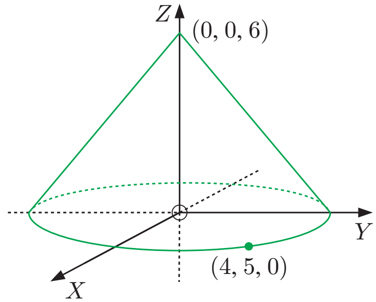
i Find the coordinates of M.

ii Find the length of [MD].

iii Hence find the surface area of the pyramid.

4 Find the volume and surface area of a rectangular-based pyramid with base coordinates O(0, 0, 0), A(10, 0, 0), B(10, 18, 0), and C(0, 18, 0), and apex (5, 9, 12).

5 The base of a cone lies in the X-Y plane, and is centred at the origin.
The point $(4, 5, 0)$ lies on the edge of the base, and the apex of the cone is $(0, 0, 6)$.



a Find the base radius of the cone.
b Find the exact volume of the cone.
c Find the slant height of the cone.
d Hence find the surface area of the cone.

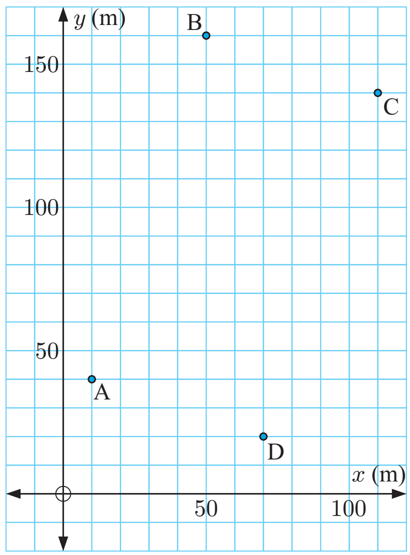
6 The point $(-4, -6, 10)$ lies on the surface of a sphere with centre $(2, 3, -1)$.
a Find the radius of the sphere.
b Hence find the volume of the sphere.

7 A sphere has diameter [PQ], where P is $(-1, 1, 2)$, and Q is $(-5, 7, -8)$.
a Locate the centre of the sphere.
b Find the radius of the sphere.
c Hence find the volume and surface area of the sphere.

8 The base of a cylinder lies in the X-Y plane, and is centred at $(1, -3, 0)$. The point $(-2, -2, 0)$ lies on the edge of the base, and the cylinder has volume 40π units^3.
a Find the height of the cylinder.
b The point $(3, k, 2)$ lies on the curved surface of the cylinder. Find the possible values of k.
c Find the surface area of the cylinder.

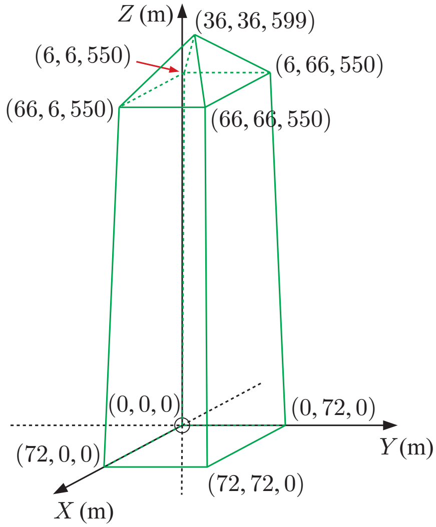
9 A square-based pyramid has coordinates A$(4, 0, 2)$, B$(-1, 3, 2)$, C$(1, -5, 2)$, and D. The apex lies directly above the centre of the base, and the pyramid has surface area 210 units^2.
a Find the coordinates of D.
b Find the height of the pyramid, correct to 1 decimal place.
c Hence find the volume of the pyramid.

10 The map for a park is shown alongside, where each grid unit represents 10 m.
A large tent is to be constructed on the park for a festival. It will be a rectangular-based pyramid with base ABCD indicated on the map. The apex of the tent is 15 m above the centre of the base.



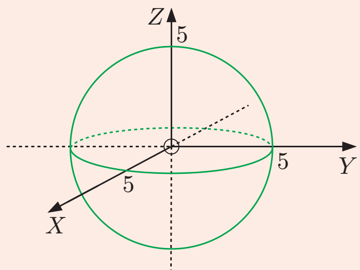
a Suppose ground level has Z-coordinate 0. Find the 3-dimensional coordinates of:
  i each corner of the base
  ii the apex of the tent.
b Find the volume of air inside the tent.
c Find the amount of material needed for the tent. Do not include the floor.

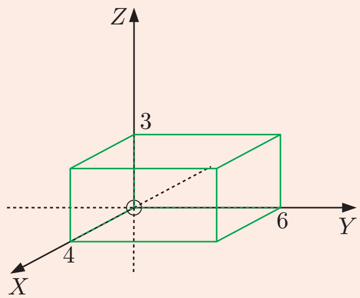
11 The Ping An Finance Centre in Shenzhen, China, is the fourth tallest building in the world.
Find the volume of this building.

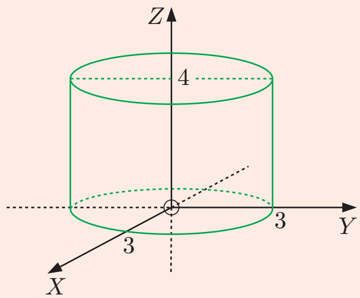


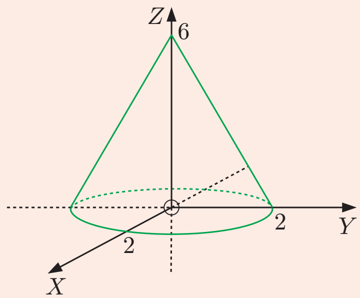
DISCUSSION

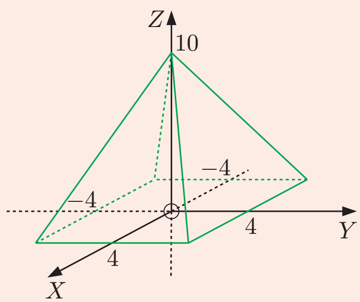
Describe an algebraic test to determine whether a given point (a, b, c) lies inside each solid:

- 

- 

- 

- 

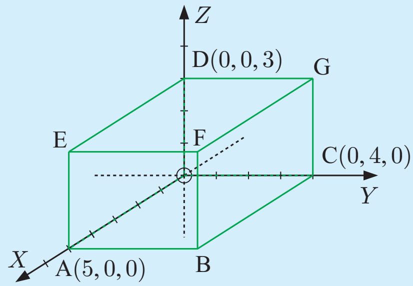
- 

C TRIGONOMETRY

The trigonometry techniques we have studied can also be applied to figures in 3-dimensional space.

Example 3 **Self Tutor**

Find the angle between the line segment [EC] and the base plane ABCO.



The required angle is $\widehat{ACE}$.

Now $\quad AE = 3$ units

and $\quad AC = \sqrt{(0-5)^2 + (4-0)^2 + (0-0)^2}$

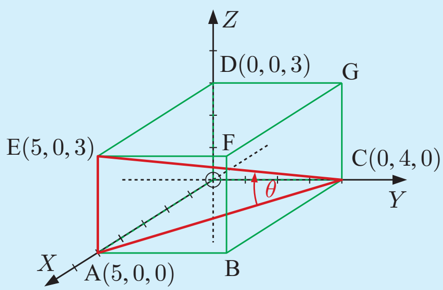
$\quad\quad\quad = \sqrt{(-5)^2 + 4^2}$

$\quad\quad\quad = \sqrt{41}$ units

$\therefore \; \tan\theta = \frac{3}{\sqrt{41}}$

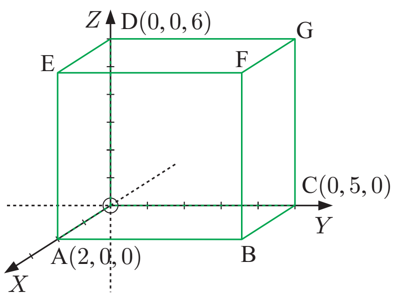
$\therefore \; \theta = \tan^{-1}\left(\frac{3}{\sqrt{41}}\right) \approx 25.1°$

The angle is about 25.1°.



EXERCISE 10C

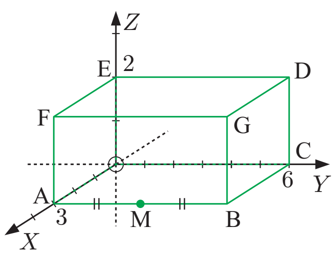
1



Find the angle between the following line segments and the base plane ABCO:

a [BE] **b** [AG]

2



Find:

a the midpoint M of [AB]

b $\widehat{ADO}$

c $\widehat{BMD}$.

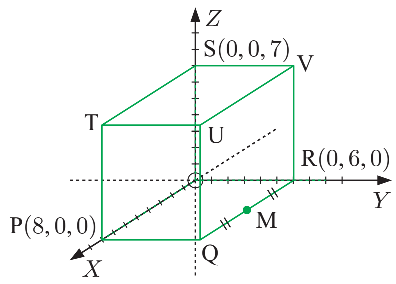
3 M is the midpoint of [QR].

a Find the coordinates of M.

b Find the measure of $Q\widehat{M}T$.

c Find the angle between the following line segments and the base plane:

i [QS] **ii** [TM]



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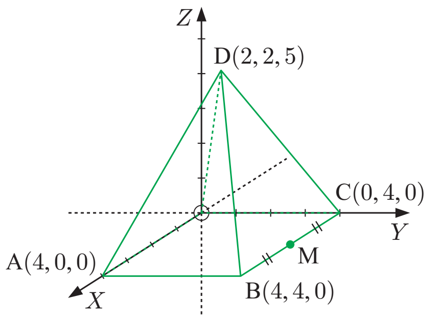


a State the coordinates of M.

b Find the angle between the following line segments and the base plane of the pyramid:

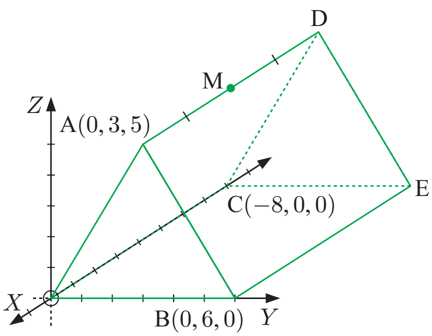
i [DM] **ii** [DA]

5 **a** State the coordinates of M.

b Find the angle between the following line segments and the base plane:

i [AE] **ii** [MB]

c Find the angle $A\widehat{B}M$.



Example 4 **Self Tutor**

Given P(1, 4, 5), Q(−2, 0, 1), and R(0, −2, 6), find the measure of $P\widehat{Q}R$.

$PQ = \sqrt{(-2-1)^2+(0-4)^2+(1-5)^2} = \sqrt{(-3)^2+(-4)^2+(-4)^2} = \sqrt{41}$ units

$PR = \sqrt{(0-1)^2+(-2-4)^2+(6-5)^2} = \sqrt{(-1)^2+(-6)^2+1^2} = \sqrt{38}$ units

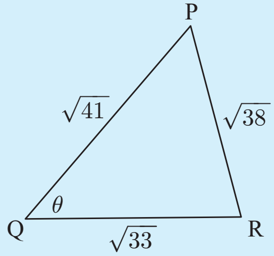
$QR = \sqrt{(0--2)^2+(-2-0)^2+(6-1)^2} = \sqrt{2^2+(-2)^2+5^2} = \sqrt{33}$ units

By the cosine rule,

$$\cos\theta = \frac{(\sqrt{41})^2+(\sqrt{33})^2-(\sqrt{38})^2}{2\times\sqrt{41}\times\sqrt{33}}$$

$$\therefore\ \cos\theta = \frac{41+33-38}{2\times\sqrt{41}\times\sqrt{33}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{36}{2\times\sqrt{41}\times\sqrt{33}}\right) \approx 60.7°$$



So, $P\widehat{Q}R$ is about 60.7°.

6 Consider the points $A(-1, 0, 2)$, $B(4, 1, 1)$, and $C(-2, 2, 0)$.

a Find the length of:

i [AB] **ii** [AC] **iii** [BC].

b Find the measure of $A\widehat{B}C$.

7 Find the measure of $P\widehat{Q}R$ given:

a $P(0, 2, -2)$, $Q(3, -1, -4)$, $R(4, 0, -1)$ **b** $P(-3, 2, 1)$, $Q(5, 0, 1)$, $R(-3, -1, 2)$.

8 Consider the points $A(-1, -5, 2)$, $B(0, -2, 3)$, and $C(4, 0, -2)$.

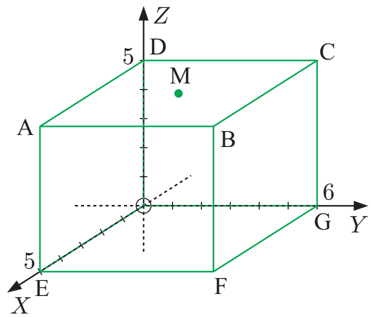
a Find the measure of $B\widehat{A}C$. **b** Hence find the area of triangle ABC.

9 Suppose A is $(1, 2, 3)$, B is $(2, 5, k)$, C is $(5, 1, 0)$, and $B\widehat{A}C = 60°$. Find k.

10 **a** Find the coordinates of M, the centre of the face ABCD.

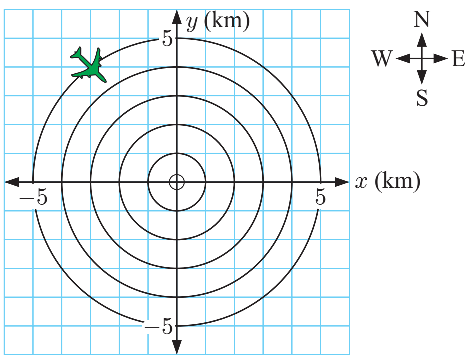
b Calculate the angle:

i $O\widehat{E}M$ **ii** $A\widehat{M}G$ **iii** $O\widehat{M}F$.



11 Answer the **Opening Problem** on page **234**.

12 The radar from an airport's control centre is shown alongside. The grid units are kilometres.
An aeroplane with altitude 500 m appears on the radar at $(-3, 4)$. It is approaching a runway 1 km east of the control centre.



a Find the 3-dimensional coordinates of the plane.

b How far is the plane from the control centre?

c Find the true bearing of the runway from the plane.

d Find the angle of the plane's descent.

13 Explorers Gabriel, Jack, and Malina are located at $(2, 3, 1)$, $(-1, -3, 1.2)$, and $(0, 4, 0.7)$ respectively. The units are kilometres and $Z = 0$ represents sea level.

a How far apart are Jack and Malina?

b Write the explorers in order of altitude, from highest to lowest.

c Find the angle of:

i elevation from Gabriel to Jack

ii depression from Gabriel to Malina.

THEORY OF KNOWLEDGE — EUCLID'S POSTULATES

Euclid was one of the great mathematical thinkers of ancient times. He founded a school in Alexandria during the reign of Ptolemy I, which lasted from 323 BC until 284 BC.

Euclid's most famous mathematical writing is a set of 13 books called *Elements*. It is the first attempt to create a complete, systematic study of geometry as pure mathematics, and has been a major source of information for the study of geometric techniques, logic, and reasoning.

Elements is sometimes regarded as the most influential textbook ever written, consisting of definitions, postulates, theorems, constructions, and proofs. It was first printed in 1482 in Venice, making it one of the first mathematical works ever printed.

The foundation of Euclid's work is his set of postulates, which are the assumptions or **axioms** used to prove further results. Euclid's postulates are:

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. **Parallel postulate**: If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

1 Can an axiom be proven? Is an axiom necessarily true?

2 Consider the first postulate.

- **a** What is a straight line? How do you know that a line is straight?
- **b** Is straightness more associated with shortest distance or with shortest time? Does light travel in a straight line?
- **c** Is straightness a matter of perception? Does it depend on the reference frame of the observer?

VIDEO

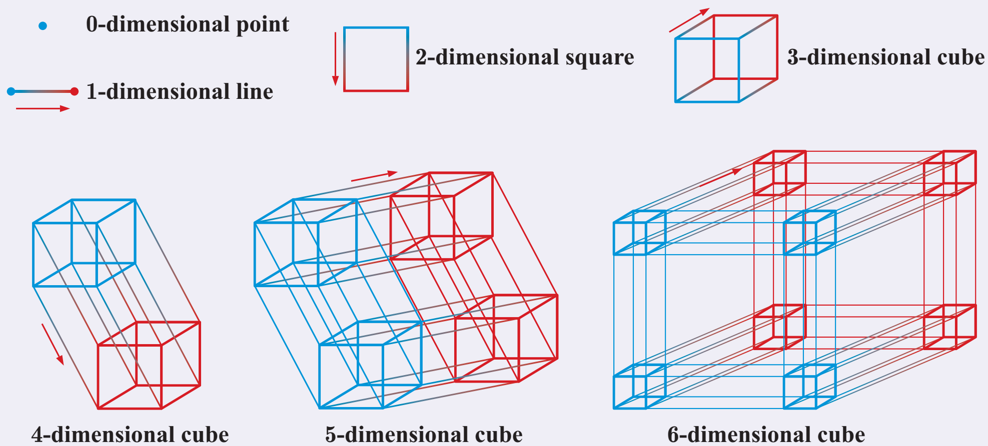
3 For hundreds of years, many people believed the world to be flat. It was then discovered the world was round, so that if you travelled for long enough in a particular direction, you would return to the same place, but at a different time.

- **a** How do we define *direction*?
- **b** Is a three-dimensional vector sufficient to describe a direction in space-time?
- **c** Can any straight line segment be extended indefinitely in a straight line?

4 Comment on the definition:

A straight line is an infinite set of points in a particular direction.

5 We are used to representing 3-dimensional objects such as a cube on 2-dimensional paper. This process can be extended to give visual representation to "cubes" in higher dimensions:



If the extra dimensions do not represent physical space, is there purpose to giving them physical representation?

6 Can the universe be completely described by a finite-dimensional space?

REVIEW SET 10A

1 On separate axes, plot the points:

a $(3, 1, 0)$ **b** $(-4, 0, 2)$ **c** $(2, -3, -1)$

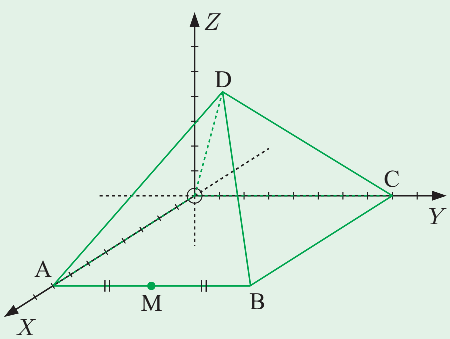
2 For each pair of points, find:

i the distance PQ **ii** the midpoint of [PQ].

a $P(1, -2, 0)$, $Q(-3, -6, 2)$ **b** $P(-3, 1, 6)$, $Q(-2, -7, 1)$

3 Suppose A is $(-2, 5, 1)$, B is $(3, 5, -3)$, and C is $(0, -1, 2)$. Determine whether triangle ABC is scalene, isosceles, or equilateral.

4



A square-based pyramid has base coordinates $O(0, 0, 0)$, $A(8, 0, 0)$, $B(8, 8, 0)$, and $C(0, 8, 0)$. The apex of the pyramid is $D(4, 4, 6)$. M is the midpoint of [AB].

a Find the volume of the pyramid.

b Find the coordinates of M.

c Find the length MD.

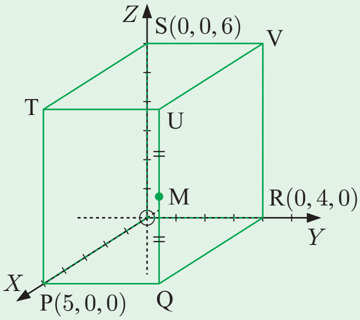
d Hence find the surface area of the pyramid.

e Find the measure of $\widehat{MDB}$.

5 The base of a hemisphere lies in the X-Y plane, and is centred at the origin. The point $(2,\ -5,\ 0)$ lies on the edge of the base.

a Find the radius of the hemisphere.

b Find the volume and surface area of the hemisphere.

6 

In this rectangular prism, M is the midpoint of [UQ].

a Find the coordinates of M.

b Find angle $\text{U}\widehat{\text{M}}\text{S}$.

c Find the angle between the following line segments and the base plane:

i [PV] **ii** [OM]

7 Given $\text{A}(-1,\ 2,\ 5)$, $\text{B}(0,\ 3,\ 1)$, and $\text{C}(2,\ -4,\ 0)$, find the measure of $\text{B}\widehat{\text{A}}\text{C}$.

8 Consider the points $\text{A}(8,\ -7,\ 2)$, $\text{B}(-1,\ k,\ 8)$, and $\text{C}(1,\ -3,\ 2)$. $\text{CA} = \text{CB}$ and $k > 0$.

a Find k.

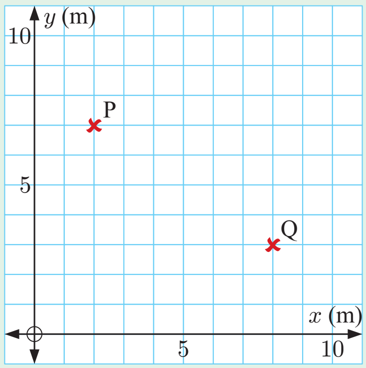
b For the circle centred at C and passing through A and B, find the area of the minor sector CAB.

9 An archaeological dig site has been divided up using grid lines at 1 metre intervals, to make it easier to describe locations on the site.
Fossils have been found at P, 2.5 m underground, and at Q, 2.9 m underground.

a Suppose ground level has Z-coordinate 0. Find the 3-dimensional coordinates of each fossil.

b Find the distance between the fossils.

c Find the angle of depression from P to Q.



REVIEW SET 10B

1 For each pair of points, find:

i the distance AB **ii** the midpoint of [AB].

a $\text{A}(-3,\ 0,\ 5)$, $\text{B}(-1,\ 6,\ 4)$

b $\text{A}(-7,\ 4,\ 6)$, $\text{B}(-2,\ 1,\ -1)$

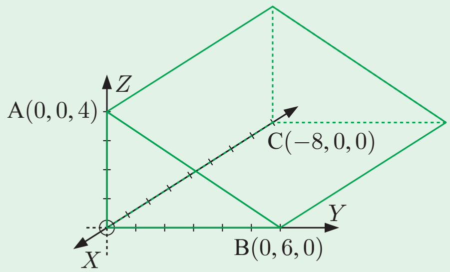
2 Suppose P is $(-5,\ 0,\ 1)$, Q is $(-2,\ -2,\ 2)$, and R is $(-1,\ 5,\ -1)$.

a Show that triangle PQR is right angled.

b Find the measure of $\text{P}\widehat{\text{Q}}\text{R}$.

3 The distance from $(4,\ -2,\ 1)$ to $(1,\ 3,\ k)$ is 8 units. Find the possible values of k.

4 Consider the triangular-based prism shown.



- **a** Find the volume of the prism.
- **b** Find the length of [AB].
- **c** Hence find the surface area of the prism.

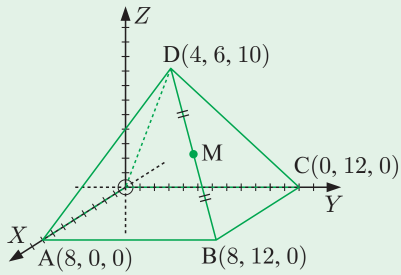
5 A sphere has diameter [PQ], where P is $(4, -2, 3)$ and Q is $(-6, 2, -5)$.

- **a** Find the coordinates of the centre of the sphere.
- **b** Find the radius of the sphere.
- **c** Hence find the volume and surface area of the sphere.

6 Consider the points $P(-2, 1, 3)$, $Q(0, -1, 4)$, and $R(-3, 2, 0)$.

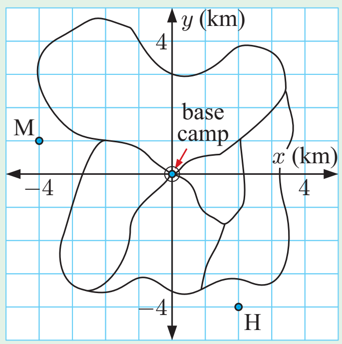
- **a** Find the measure of $P\widehat{R}Q$.
- **b** Hence find the area of triangle PQR.

7 In the rectangular-based pyramid alongside, M is the midpoint of [BD].



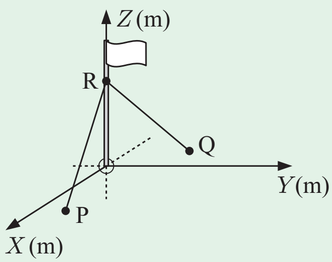
- **a** Find the coordinates of M.
- **b** Find $A\widehat{D}C$.
- **c** Find the angle between the following line segments and the base plane:
  - **i** [DA]
  - **ii** [MC]

8 On the terrain map shown, the grid units are kilometres. The hiker is at a viewing platform H, 200 m above the base camp O.



- **a** State the 3-dimensional coordinates of the hiker.
- **b** How far is the hiker from the base camp?
- **c** From the viewing platform, the hiker can see the summit of a mountain at M. The mountain top is 500 m above base camp level.
  - **i** Find the 3-dimensional coordinates of the mountain top.
  - **ii** Find the distance between the hiker and the mountain top.
  - **iii** Find the angle of elevation from the hiker to the mountain top.

9 A flagpole is located on the Z-axis. It is supported by wires fixed to the ground at $P(3, 1, 0)$ and $Q(-1, 2, 0)$. The wires meet at R, 3 m up the flagpole.



- **a** Find the coordinates of R.
- **b** Find the angle each wire makes with the flagpole.
- **c** Find the angle at which the wires meet.